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A ... of Checking Upon Accuracy in Addition
JOHN R. CLARK and ... VINCENT 65
Ch... Junior High School Mathematics ... -
... WORTHINGTON 7[illegible]
... of Geometry ... HEFFERMAN 78
... High School ... JOPKINS 81
The ... Committee ... -
... B... 86
Con... - 99
Effe... Drills in Arithmetic -
... A. I. MESSICK 104
Th... SIDNEY WALDE 110
... B. MCCAIN 112
T... THEODORE STRONG 115
N... - 120
Ne... - 123
Me... Teachers of Mathematics - 124

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THE MATHEMATICS TEACHER

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A STUDY OF THE EFFECT OF CHECKING UPON ACCURACY IN ADDITION

JOHN R. CLARK and E. LEONA VINCENT

Purpose: The purpose of this investigation was to determine the effect, of twenty days of practice in checking in single column addition, upon speed and accuracy in single column addition and upon efficiency in more complex addition and in multiplication.

Procedure: The fifth and sixth grades of the Lincoln School, 87 children in all, were given an initial test comprising 25 single column additions in which no digit less than four was repeated. The time allowed was four minutes. The Courtis Standard Supervisory Test (addition and multiplication sections) was also given. On the basis of mental age, intelligence quotient, and initial addition test scores, the pupils were divided into two equivalent groups. After 20 days of practice in adding, one group using the technique of checking to be described later, duplicate forms of the initial tests were repeated.

A score combining speed and accuracy. In scoring the initial test as well as the daily practice papers it became necessary, of course, to take into consideration both speed and accuracy. An acceptable computational index which adequately combines the factors of speed and accuracy has not as yet been discovered, nor can it be until the casual relationships between them are known.

We would grant that the child who can add 15 columns correctly in four minutes has a greater efficiency than the child who adds only 12, or that if two children add 12 columns each and one has 11 correct, while the other has only 9, the former is the more efficient. But whether the child who attempts 10 columns and gets 9 right is more efficient than the one who attempts only 6, but gets them all right is an open question.

If we consider only the *percentage* of correct solutions, we give the child who does one right out of one attempt quite as much credit as the child who does 10 right out of 10 attempts. On the other hand if we consider simply the actual number right

we give the child who gets only 10 correct out of, say, 20 attempts more credit than the child who attempts only 9 but gets them all correct.

If, however, we multiply the percent right by the actual number right, we have such an index that the child who does 10 right out of 20 attempts gets a score of 500 which is the same score that another child gets who does only 5, but gets them all right. Again, 12 right out of 16 attempts gives a score of 900 which is comparable with 9 out of 9. Six out of 7, however, gets a score of 514 which is slightly superior to 5 out of 5, or 500. Eighteen out of 20 gives 1,620, which is a little more than the 1,600 resulting from a computation of the index for 16 out of 16. Whether this is as it should be we cannot say, but it seems the best available measure at present. Accordingly, the index used throughout this study is represented by the formula

$$I = \frac{100R^2}{A}$$

in which I is the index, R is the number right, and A is the number attempted.

Practice: One group was given instruction in the technique of checking. In the daily practices the children in this group were to add each column in one direction and write the answer, then in the opposite direction and write that answer. If the two answers agreed they were to go on to the next column; if not, to add again, and again, until two agreeing answers were obtained. The other group was instructed to add each column only once throughout the practice, this being the method children generally use in practice work.

The two groups were given four minutes of practice on each of 20 successive school days. The daily practice material consisted of single columns of nine digits in which no digit less than four was repeated. In the final tests, the pupils in both groups were told that only one answer would be expected to each problem, so there was no visible evidence of checking.

Results: The amount of variability in the addition scores from day to day made it quite evident that the scores of any single day or of any one test could not be considered sufficiently reliable to provide a measure of the results. For this reason the

differences in scores for the two groups in the final addition test has not been used, but rather the difference between the medians of the daily medians of the two groups. Table I shows the daily medians and the median of these medians for each group. Two things stand out:

TABLE I

SHOWING MEDIAN INDEX SCORES FOR BOTH GROUPS FROM THE INITIAL TEST THROUGH THE DAILY PRACTICES TO THE FINAL TEST

	Checking Group	Non-Checking Group
Initial Test	1163	1170
1st Practice	1033	890
2nd "	933	1333
3rd "	1250	916
4th "	1014	1016
5th "	975	900
6th "	1233	1183
7th "	1237	1025
8th "	1316	1175
9th "	1425	1260
10th "	1210	963
11th "	1150	983
12th "	1233	950
13th "	1375	1040
14th "	1200	980
15th "	900	1075
16th "	1380	1050
17th "	1250	960
18th "	1080	1050
19th "	1200	1189
20th "	1116	1100
Final Test	1350	1316
Median	1225	1050

(1) There is marked variability from day to day not only of the individual scores (which are not shown), but of the median scores for each group. (2) Our measure of computational efficiency (the index score) shows the checking group to be somewhat superior.

Not only the variability, and the superiority of the checking scores, but the small amount of net improvement resulting from practice are to be noted. We cannot understand what this means until we study the factors of speed and accuracy separately.

Table II shows the median number of attempts for the initial test, final test, and daily practices for each group. The figures for the checking group represent the actual number of additions made, not the number of different columns attempted.

TABLE II

SHOWING THE DAILY MEDIAN ATTEMPTS FOR CHECKING AND NON-CHECKING GROUPS

	Checking Group*	Non-Checking Group
Initial Test	14.0	17.0
1st Practice	12.0	12.1
2nd "	11.3	12.8
3rd "	14.4	13.3
4th "	12.8	13.4
5th "	11.6	12.5
6th "	14.0	15.3
7th "	13.9	14.5
8th "	15.6	14.8
9th "	16.5	17.1
10th "	13.7	14.5
11th "	15.0	14.8
12th "	16.5	19.5
13th "	14.5	16.7
14th "	15.4	15.0
15th "	12.3	16.6
16th "	16.0	17.2
17th "	14.6	14.8
18th "	14.0	17.0
19th "	13.5	16.7
20th "	14.3	14.5
Final Test	17.0	18.4
Median of Medians	14.15	14.9

In speed the non-checking group exceeds the checking group. It also varied more in its performance from day to day. The median of the medians of number attempted for the non-checking group was 14.9 as against 14.15 for the checking group. The non-checking group attempted more additions per unit of time than did the checking group.

Table III shows the median number of rights for the initial test, final test, and daily practices for each group.

The non-checking group still varies more from day to day, but is in number of rights exceeded by the checking group. The median of the daily median number of rights for the checking group is 13.1; for the non-checking group 12.6. *Although the checking group has less speed, it has greater accuracy than the non-checking group.* Further evidence of this follows.

It was thought desirable to know something of the actual totals for each group; two samplings were taken. In one case 1,000 added columns were chosen at random from among the papers

*These figures represent the medians of the actual number of additions made by each child each day, not the number of different columns attempted.

of the non-checking group. In the same way 1,000 columns were chosen from the papers of the checking group. Of the 1,000 non-checking columns 850 were correct; of the 1,000 checking columns 932 were correct, a difference of 82. This one would expect since each column of the latter group had been checked.

TABLE III

SHOWING DAILY MEDIAN RIGHTS FOR CHECKING AND NON-CHECKING GROUPS

	Checking Group	Non-Checking Group
Initial Test	13.0	14.5
1st Practice	10.8	10.4
2nd "	10.3	10.6
3rd "	13.3	11.7
4th "	10.7	12.3
5th "	10.8	10.2
6th "	12.7	13.3
7th "	12.6	12.0
8th "	14.8	13.0
9th "	14.8	14.8
10th "	12.5	11.2
11th "	13.6	11.6
12th "	14.6	15.9
13th "	14.2	12.5
14th "	14.0	12.6
15th "	10.8	12.8
16th "	14.5	14.0
17th "	13.8	11.0
18th "	13.2	13.6
19th "	12.5	14.0
20th "	12.5	12.7
Final Test	14.7	16.2
Median of Medians	13.1	12.6

In this 1,000 columns from the checking group, it is interesting to note that 93.2 per cent had agreeing answers which were right; 6.5 percent of the columns had been added three times or more, and yet did not have correct answers; 4.8 percent had been added four times or more, but were still wrong; .8 percent had been added five times without success; .3 percent of the columns had been left without agreeing answers.

In order to determine in another way the relationship between speed and accuracy in the two groups, 150 papers were chosen at random from each group. On these papers the non-checking group had attempted 2,229 columns with an accuracy of 77.9 percent, while the checking group had attempted only 970 col-

umns, but with an accuracy of 90.1 percent. In this sampling, which represents the same working time for each group, the checking group added less than half as many different columns, but with 12.2 percent greater accuracy. It is a well-known fact that any increase in accuracy beyond approximately 75 percent is very difficult to secure, and that although 100 percent accuracy is the goal, anything beyond 90 percent is almost never attained. Whether a cut in speed of over half is a fair price to pay for it is a question.

It is again interesting to note that while the checking group was getting answers for these 970 examples, a total of 2,138 actual additions were made. That is, 2,138 "column-additions" were made in doing the 970 examples. It would be natural to suppose that for these the same percent of accuracy would maintain as had maintained for the actual additions of the non-checking group, viz., 77.9 percent. This was not true, however, for of the 2,138 additions attempted by the checking group, 1,837, or 85.9 percent, were right. It is probably true that the people who check have in the beginning an awareness of the fact that their answers must agree and hence add more carefully in the first place.

Effect upon more complicated addition and upon multiplication. Does practice in checking single columns increase pupils' accuracy in more complicated addition? Does it spread to multiplication? To secure some evidence on this, a second form of the Courtis Standard Supervisory Test was given. The spread of the effect of checking single column additions to the kind of addition and multiplication involved in this test was measured. The gain between the first and last Courtis tests in addition was for the checking group 145 points (index score) of a possible 800 points, and for the non-checking group 64 points. In multiplication the gain for the checking group was 15 points of a possible 500, and for the non-checking group 118 points. Assuming the reliability of such measures, it would seem that practice in checking single column additions produces more improvement in complicated addition than does practice in single column addition without checking. On the other hand, less improvement in multiplication was made by the checking than by the non-checking group.

Conclusions

1. On the assumption that our index score gives a fairly satisfactory weight to both speed and accuracy, the practice of checking addition of single columns results in a greater efficiency in addition. The median of the 22 daily median index scores for the checking group was 1,225, for the non-checking group 1,050.

2. If the two groups be measured by the number of additions attempted in a given time there is an advantage in favor of the non-checking group. The median of the daily median attempts was for the checking group 14.2 (total additions, not total columns), and for the non-checking group 14.9.

3. If, however, the two groups be compared on the basis of the number of correct additions (not the number of separate columns) the advantage is with the checking group. The median of the daily median rights was for the checking group 13.1, for the non-checking group 12.6. The non-checking performed more actual additions per unit of time, but got fewer correct answers.

4. In a random sampling of over 2,000 column additions the non-checking group was found to have an accuracy of 77.9 percent. In a sampling which represented the same working time the checking group had added less than half as many columns, but with an accuracy of 90.1 percent.

5. This increase in accuracy so difficult to get under ordinary teaching conditions, may justify the substitution of this method of checking for the methods now generally used in drill work in arithmetic. If so, it will be necessary to adjust the various types of practice material and test norms to the new standards.

6. Practice in checking single column additions seems to transfer to more complex addition, but not to multiplication. On the other hand, practice in single column addition without checking seems to transfer to multiplication but not to more complex addition. The writers do not feel that the data for this statement justify more than mention.

CURRENT PRACTICE IN JUNIOR HIGH SCHOOL MATHEMATICS

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Since the report of the National Committee on Mathematical Requirements, under the auspices of the Mathematical Association of America, made its appearance we have been more certain of the content of Junior High School Mathematics, although the committee gave no order of topics with specific time allotments of each. A question which arose in my mind last spring was, "What are the Junior High Schools throughout the country definitely teaching in Mathematics?" I, accordingly, wrote to Dr. Glass, of the State Department of Pennsylvania, for a list of twenty-five of the best Junior High Schools in Pennsylvania, exclusive of Philadelphia. Philadelphia was excluded as most of the members of my class were already teaching in Philadelphia and were acquainted with the Philadelphia course of study. We received a directory of the Junior High Schools of Pennsylvania with twenty-five of the best checked by Dr. Glass. Books on Junior High Schools were then scanned for names of Junior High Schools, but with little success as they referred mainly to large school systems and we desired to get in touch with the mathematics departments. We found in the ninth year book of the National Association of Secondary-School Principals, in the directory of members, about forty-five names of Junior High Schools throughout the country.

With the assistance of two members of the class a study was made by sending out a questionnaire and thirty-three replies from Junior High Schools in sixteen scattered States excluding Pennsylvania and eighteen replies from Junior High Schools in Pennsylvania were received. In order that the reader might have an idea of the territory covered the States are as follows: Arkansas, California, Colorado, Illinois, Indiana, Iowa, Kansas, Massachusetts, Michigan, Minnesota, Missouri, Nebraska, New Jersey, New York, Ohio, Wisconsin. In this study *Pa.* was used for Pennsylvania Junior High Schools and *U. S.* for Junior High Schools outside of Pennsylvania.

TABLE I: PERIODS PER WEEK DEVOTED TO MATHEMATICS

	Number of Periods Devoted to Math. in 18 Pa. J. H. S.					Number of Periods Devoted to Math. in 33 U. S. J H. S.				
	Required		Elective			Required		Elective		
Periods per week	4	5	4	5		4	5	4	5	
7th grade	3	15	0	0	18	4	29	0	0	33
8th grade	4	14	0	0	18	5	28	0	0	33
9th grade	3	13	0	2	18	3	18	0	12	33

We thus see from the above table that mathematics is required in grades 7 and 8 in all 51 J. H. S. which are included in this study and that about 90 percent have mathematics five periods per week. Most of these are periods of forty-five minutes each. In the ninth year there seems to be a tendency, especially in the western Junior H. S. to allow a choice between algebra, business arithmetic and occasionally general mathematics.

TABLE II: NAME OF MATHEMATICS TEXT BOOK USED

	Frequency Pa.	U. S.	Author	Text Book
Grade 7				
*of	0	1	Betz	Geometry for J. H. S.
Frequency	0	1	Boylan Smith	City Arithmetic
5 or more	0	2	Chadsey-Smith	Arithmetic
	2	2	Gugle	Modern Junior Math. Bk I
	2	1	Hamilton	Arithmetic
	*3	2	Hart	J. H. S. Math. Bk. I
	*2	3	Hoyt and Peet	Arithmetic
	0	1	Milne	Algebra
	*2	8	Schorling & Clark	Modern Mathematics for 7th School Year
	*3	2	Stone	J. H. S. Math. Book I
	1	3	Stone & Millis	Advanced Arith. Revised
	*4	1	Taylor & Allen	J. H. S. Math. Bk. I
	0	2	Thorndike	Arithmetic
	0	2	Vosburgh & Gentleman	J. H. S. Math. Bk. I.
	*1	6	Wentworth, Smith & Brown	J. H. S. Math. Bk. I.
	1	0	Hazelton, Penna.	City Arithmetic
Grade 8	0	1	Barker	Arithmetic
	0	1	Betz	Introductory Algebra
	0	1	Betz	Geometry for J. H. S.
	0	1	Boylan Smith	City Arithmetic
	0	2	Chadsey Smith	Arithmetic
	0	1	Failor	Geometry
	2	2	Gugle	Modern Jr. Math. Bk. II
	2	2	Hamilton	Essentials of Arith. Revised
	1	0	Hamilton	Standard Arithmetic
	*3	2	Hart	J. H. S. Math. Bk. II
	0	1	Hawkes, Luby, Touton	First Year Algebra

	1	0	Hazelton, Pa.	City Arithme.ic
	0	1	Hertz & Brantz	Arithmetic
	2	2	Hoyt & Peet	Arithmetic
	0	1	Milne	Algebra
	0	1	Rowe	Jr. Arith. Bookkeeping I & II
	*1	8	Schorling & Clark	Modern Mathematics
	1	0	Schultze	Elements of Algebra
	*3	2	Stone	J. H S. Math. Bk. II
	1	3	Stone & Millis	Advanced Arithmetic
	4	0	Taylor & Allen	J. H. S. Math. Bk. II revised
	0	2	Thorndike	Arithmetic
	0	1	Vosburgh & Gentleman	J. H. S. Math. Bk. II
	*1	6	Wentworth, Smith & Brown	J. H. S. Math. Bk. II
Grade 9				
	0	1	Breslich	1st Yr. Math. for Secondary Schools
	0	1	Durrell	Algebra Bk. I
	1	1	Durrell & Arnold	Algebra
	1	1	Edgerton & Carpenter	1st course in Algebra
	2	2	Ford & Ammerman	1st course in Algebra
	2	1	Gugle	Modern Junior Math. Bk III
	1	1	Hart	Elementary Algebra
	*3	6	Hawkes, Luby, Touton	First Course in Algebra
	1	0	Nyberg	Algebra
	2	1	Rugg & Clark	Fundamentals of H. S. Math
	*1	5	Schorling & Clark	Modern Algebra
	0	4	Schorling & Reeve	General Mathematics
	1	0	Schultze	Elements of Algebra
	0	1	Stone	J. H. S. Math. Bk. III
	1	0	Tavlor & Allen	J. H. S. Math. Bk. III
	*5	4	Wells & Hart	Modern 1st yr. Algebra
	0	1	Wenemeyers	Industrial Mathematics
	*1	4	Wentworth, Smith & Brown	J. H. S. Math. Bk. III
	1	2	Finney & Brown	Modern Business Arithmetic
	1	0	Smith	Business Arithmetic
	0	1	Sutton & Lennes	Business Arithmetic
	*3	2	Van Tuyl	Commercial Arithmetic
	0	1	Walsh	Commercial Arithmetic

If we study Table 2 carefully we notice the following: (1) There is no one or two text books used by many of the schools—16 different books in grade seven, 24 in grade eight and 23 in grade nine; (2) the number of text books in each grade exceeds the number of schools studied. (This is caused by some schools using as many as three books in one grade); (3) Over half of the books in grades seven and eight are recently published J. H. S. mathematics text books. The books in grade nine are of three types—algebra, general mathematics or industrial and business arithmetic—for the three courses Academic, General and Commercial.

TABLE III: EXTENT TO WHICH NUMERICAL TRIGONOMETRY IS TAUGHT

Grade	7	8	9
YES			
Pa.	0	2	8
U. S.	2	6	13
NO			
Pa.	18	16	10
U. S.	31	27	20
Percent of Total Yes	4%	16%	41%
Percent of Total No	96%	84%	59%

From the table we see a tendency to teach Numerical Trigonometry in a few eighth grades and in almost half of the ninth grades. One school did not understand the meaning of numerical trigonometry and confused it with the regular course in the twelfth grade.

TABLE IV: EXTENT TO WHICH AN INTRODUCTORY COURSE IN DEMONSTRATIVE GEOMETRY IS TAUGHT

	YES							NO	% not teaching it
	1-2 year	10 weeks	5 weeks	4 weeks	2 weeks	1 hr. week	Very little		
Pa.	0	1	1	0	1	0	2	13	72%
U. S.	2	0	1	3	1	1	3	22	67%

Among the interesting replies on this question was that of Professor William Betz, head of the Mathematics Department of the Washington J. H. S., Rochester, N. Y., who says, *"No, absolutely impossible!!"* No doubt the percentage of those not teaching Demonstrative Geometry could be increased at least ten points in the above table to allow for those who still confuse Demonstrative Geometry with Intuitional Geometry.

TABLE V: EXTENT TO WHICH VARIOUS OTHER TOPICS ARE TAUGHT

	Pa.		U. S.	
Topics	Yes	No	Yes	No
Logarithms	4	14	5	28
Slide Rule	5	13	5	28
Statistics	12	6	19	14
Cone	14	4	26	7
Pyramid	14	4	25	8
Sphere	15	3	25	8
Frustum	5	13	9	24

From the above table we see that logarithms, slide rule and frustum are rarely taught in the J. H. S. throughout the country, while statistics is gradually working its way into the course of study.

VI. AMOUNT OF TIME IS GIVEN EXCLUSIVELY TO ABSTRACT DRILLS IN THE FUNDAMENTALS

The most frequent replies to this question were "from five to ten minutes daily. "Seventeen of the thirty-three replies outside of Pennsylvania an dfourteen of the eighteen in Pennsylvania were of this type. This is in accordance with the best practise recommended by leading mathematics teachers throughout the country. Other replies were as follows: Two-thirds of time (1); one-third of time (2); one-quarter of time (2); varies with class (3); practically no time (1); very little (1); 20 minutes per day (3); one-half of time (4); 30 minutes per week (1); 2 hours per week (2).

TABLE VII: METHODS OF TEACHING INTEREST

	Pa.		U. S.		Total	
Topic	Yes	No	Yes	No	Yes	No
Interest Formula	13	5	30	3	43	8
Interest Graph	7	11	17	16	24	27
Interest Tables	11	7	28	5	39	12
Compound Interest Tables	9	9	17	16	26	25

The table speaks for itself. The four topics are usually taught in the seventh or eighth grades or in both, only three of the fifty-one replies stating that they were taught in the ninth year.

SUMMARY

1. Mathematics is required usually five periods per week in grades seven and eight, while the pupil is frequently permitted in grade nine to choose between algebra, general mathmatics and business arithmetic.

2. Several of the old type texts are still in use although most of the Junior High Schools are turning to the newer type of text.

3. Numerical trigonometry is gradually gaining friends, while demonstrative geometry is not making much progress. Many teachers and administrators are still ignorant of these terms.

4. Such topics as logarithms and slide rule are on the supplementary list in most text books, and teachers are not anxious to teach them. Statistics, on the other hand, is making great strides.

5. Most of the Junior High Schools studied are using interest tables, but the value of the interest graph does not appeal to them.

"MATH"

To hear some folks a talkin',
You'd think math out o' date,—
They hunt for every reason
To wipe math off the slate.

"For math is rather ancient,—
It used to be O. K.,"
Say these folks on the rampant,—
"Yes, math has had its day."

Some bright folks, educators,—
(They flunked out in their math)
"No rhyme or sense to figures,
Away!" They cry in wrath.

"For—math, we never use it,—
For math is out of style,—
We left math off the budget,—
For math is not worth while."

Forget math in your science,—
Why clutter up your pate?
A cheap, no-good appliance,—
Yes, wipe math off the slate."

WILLIAM L. HUNTER.

THE INTRODUCTION TO THE STUDY OF GEOMETRY

JOHN K. HEFFERMAN
Butler, Pa., High School

It is said that King Ptolemy, hearing of the wonders of geometry, summoned Euclid before him and commanded the latter to teach him this new science. The king was much disturbed (in mind, at least) when Euclid answered that there was no royal road to geometry. I am sure that we can exercise a little "poetic lincense" and use as our text, "There is no royal road to the teaching of geometry." So this treatise is not written as a panacea for all the academic ills connected with the subject but rather a critical study of some of the features which make up the introduction to the study of geometry.

It is evident that the experience of teachers counts for a great deal, so a few of these experiences will be set forth. The pupil, when he first appears before us, knows little of logic and perhaps cares less. He is young (fourteen or fifteen) and often his thoughts are not on geometry. But, while the mind of the average high school child is immature and occupied with concrete things such as the radio, automobile, etc., it is also eager and alert when interested. Perhaps he will ask what use is there in studying this subject, and why should he have to prove what seems to him perfectly evident. Hence here is the golden opportunity of the teacher to "sell the course" to the pupil.

Although this fact should be perfectly obvious, it is necessary that the teacher formulate (in his own mind, at least) some definite aim—that he should at least know what adaptations should take place in the pupils as a result of the study of geometry. For if the teacher himself has no conception of an objective, it is not likely that the pupils will see any.

It is often quite difficult to explain why geometry is studied, so it is perhaps best to introduce the pupil to the subject by the laboratory method, that is, the use of the compass and straight edge in the drawing of simple figures which they will use later. The use of squared paper can be introduced also. This method serves several purposes; it excites his interest, it helps to guard against slovenly figures which often leads him to erroneous con-

clusions, he gets a new insight into the utility of geometric figures in architecture, industry, etc. He is stimulated by the fact that large results can be obtained with small expenditure of time and effort, he is easily led to see the nature of a proof and to reason inductively. In general, he learns to do by doing.

The question now in your mind is, when does the student begin demonstrative geometry? But before we consider this question, let us study the nature of these preliminary theorems and definitions.

Now many of these so-called preliminary theorems are quite difficult to prove, even for more mature students. Such theorems as, *all right angles are equal, only one perpendicular can be drawn to a point in a line,* state facts so evident that the beginner can see no necessity for proving them. How then shall they be studied? The essential thing to remember is that the pupil learns to demonstrate by demonstrating. He should be guided by the teacher, with the subject matter duly simplified and cut up into small portions for him. The steps cannot be too easy at first and the teacher should not hesitate to do all the preliminary drudgery for the pupil. Leave to the child the pleasure of merely discovering the proof. Arouse in him the pleasure of attaining large results from small effort. Some of these preliminary theorems will seem more difficult than others. If the pupil is not able to see clearly the proof of these preliminary theorems, let him assume them as axiomatic. If his demonstration is a bit faulty, there should be no great cause for alarm, for these minor difficulties can easily be cleared up later. Do not dwell too long on these preliminary theorems, nor insist upon a rigorous proof. By dwelling too long, the pupil will sometimes become confused and often suspect difficulties which did not exist before. Perhaps the most harm is done by the teacher who insists upon a rigorous proof. For the pupil soon learns that only by memorizing can he satisfy his rigorous teacher. Besides making the dislike for the subject the student learns the wrong method of study. Besides making the proof simple, and assuming some as axiomatic, it is important to give them as many applications as possible. This may be rather difficult for some theorems, but the ingenious teacher should be able to devise some.

Now, as to the preliminary definitions. Shall we compel the child to learn each definition word for word so that he will be

able to give it back on demand or shall we make it more as a logical exercise? Proponents of the former plan will argue that the child is learning a new vocabulary, that it is necessary for him to become acquainted with technical terms, and that he cannot hope to improve upon the language of the text. They will further argue that indefinite and variable use of technical terms has been a source of errors and misunderstanding in many sciences. Still others will urge this method on the grounds of the "logical training" it is said to give. While I will admit that "indefinite and variable" use of technical terms has been a source of errors, in geometry most of the terms are so definite that the danger of misunderstandings will be slight. While it may be true that the formal teaching of definition can be made an exercise in logic, I doubt if it is so in the majority of cases. Again, I will admit that the student will not be able to improve upon the author's version, but what value to the student if he is able to glibly let loose a volley of words which mean little or nothing to him? To be a logical exercise, the students themselves should formulate these definitions. It is evident, that, in this latter method, the student must have a clear idea of the thing to be defined—hence, when the student himself gives the definition, he would make the study of formal definitions the end and not the starting point. Thus, they would be studied not for the idea of conveying to the students the meaning of a word or thing, but merely for the training in logic. Perhaps this method would not be applicable to all definitions but by means of numerical exercises, drawing exercises and laboratory exercises, this method is also an excellent plan to familiarize the student with technical terms. By the former method, it is not likely that the pupil will know much about technical terms, for only by actual work involving these terms can he hope to obtain any light on this phase of the subject. However, I do not believe that definitions present so great a difficulty as other phases of the introduction, nor are they so important.

APPLIED MATHEMATICS IN HIGH SCHOOL

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Much is being said and written by present day educators urging upon mathematics teachers the advantage and necessity of appealing to the child's interests through an abundance of so-called practical problems. Practical is a very narrow term, however. In order rightfully to be so called, a problem must involve a solution which satisfies a conscious need on the part of the one who solves it. We are using the term applied in a much broader and more inclusive sense, including any type of problem in which mathematics is put to direct use in modifying environment whether by the student himself or by another, the engineer, the scientist, the business man, the architect, or the surveyor.

Every teacher must decide for herself the question "Why do I teach mathematics? What do I expect the child to gain from it to repay him for his time and effort? What can he gain from it that no other course will give him?" In the minds of some, the utilitarian value is most important and those who believe this lay stress on the practical problems, seeking ways and means of bringing the mathematics of the child's life out of school into the class room, and the mathematics of the classroom into his life outside of school. Others stress the disciplinary side, having as their primary aim the training of the child in exact, logical, and organized thinking and in assuming a problem solving attitude of mind. Some who find great joy in the subject for its own sake, seek to instill in their students the same delight in the subject so they will continue it for the pleasure of accomplishment. No doubt all of these values are derived in a measure, regardless of the aim that is foremost in the mind of the teacher. In order to achieve the maximum of efficiency, however, a teacher should decide just what values she considers most important, and then aim directly at these as ends. Only in this way can the maximum good be accomplished.

What place then should be given to applied problems? To what end should they be included and how much attention should they receive? We will not all agree on this point. Contrary to

the opinion of many teachers, I am convinced that the teachers of mathematics can accomplish more, can get better results, and encourage more pupils to continue their study, by making the primary aim sentimental, that is the study of the subject for its own sake, for the pleasure found in its pursuit, than in any other way. In 1905 a French mathematical journal sent out an inquiry to well-known mathematicians asking them why they took up the study of mathematics as a specialty. The replies of those who responded indicated that their interest was due to a desire to develop the subject for its own sake rather than for its applications. If the experience of these men is representative of mathematicians in general, and no doubt it is, then mathematics is more fascinating on its own account than for what it can accomplish. The pedagogical deduction is obvious. The wise teacher will expect and encourage the same reaction on the part of her students. I consider this responsiveness on the part of my students to be the best criterion available for measuring my success as a teacher.

The deduction is, therefore, that the application of mathematics is not our primary aim in teaching the subject. Why then include it? Some advocate it as a means of motivation. This would hardly be in keeping with the statements made above, and if used primarily for that purpose would not lead to the highest appreciation of pure mathematics. True, there will always be some who cannot be led to appreciate mathematics for its own sake, and for these applied problems may serve as a means of motivation even though they are not introduced for that purpose. We do not include them purely on utilitarian grounds, for very few of the applications learned in the class room will be of practical use to the students in after life. Applied mathematics should be taught rather for the purpose of developing a better understanding and comprehension of one's environment, and appreciation of the relation of mathematics to the development of civilization and to the miracles of present day science, that the child may know that mathematics is a living, growing, dynamic, science, to which he owes the comforts and luxuries he enjoys today.

Those of us who are studying and teaching the subject, well know that it is a most important and basic element in our

modern civilization that the development of civilization is and always has been closely bound up with the world's knowledge of mathematics. We know that our industries and commerce depend upon mathematical development and that the wiping out of all traces of mathematical knowledge would cause the world to revert immediately to barbarism. In order to develop on the part of the student a deep appreciation for the subject he must be made to realize all this—that he is studying a subject that the people about him find most necessary to use in every day human activities. To this end we teachers should be on the alert for applications that are important in the world today, even bringing up occasionally those of a more technical sort that can be discussed only in a general way. For, as Strayer and Norsworthy tell us "It is conceivable that the person of culture is one who by virtue of his education has come to understand and appreciate the many aspects of social environment in which he lives."

The human mind is very curious with reference to all phases of environment. Here follows a few of the questions which have been called to my attention recently, from adults as well as children, in conversations and reading as well as in the schoolroom. Who has not had occasion to discuss these and others like them? How could MacMillan know if he reached the North Pole? How is it possible to start building a bridge from both sides of a river? Why are there so many jogs in the streets and sidewalks of Omaha? How can the fall of a river be determined? How can a scientist predict a comet or an eclipse? If a man borrows $4,000 on his home, how long will it take to clear the debt if he pays $75 a month? If a man saves five dollars each week from his salary beginning at the age of twenty-one, and invests it at five per cent interest compounded semi-annually, how much will he be worth if he retires at the age of sixty-five? How were the barrage maneuvers of the world war made possible? How can an aviator determine his elevation? How is the depth of a sea or lake determined; or the width of a river or lake at an impassable point? High school mathematics does not intend to give students the ability to solve problems of this sort, yet it should give them an insight into and and understanding of the fact that it is mathematics that makes these things possible, and so develop a respect and appreciation for its power.

There are many simple phases of applied mathematics which should find a place in the high school course. The power of the compound interest formula, both in the making of the table and the graph, is quite startling to the average student and makes an excellent lesson in thrift. Very few will make use of the slide rule outside the classroom, but to learn its use satisfies a most normal and legitimate curiosity, and it is a delight to the average student to know the many kinds of computation that can be performed with this one little instrument. The introduction of the elements of trigonometry into our geometry work gives an excellent opportunity to teach the simple applied uses of geometry and algebra. We have no desire to make surveyors and engineers of our students, but if a transit is available, its uses in the classroom are endless. Time will not permit teaching all the students in large classes how to use and care for it, but a demonstration of its use is a revelation to them. I have found it profitable when possible to take time to solve a few problems in triangulation with its aid—the height of the flag pole, the width of a street without crossing it, the height of a hill if one is available, and so on. After having had a transit for use in geometry and trigonometry for a number of years, I became more than ever convinced that it should have a definite place in our high school work. After two or three days' work with it in the classroom, my experience has been that students are on the alert to watch surveyors at work, because they understand, even though in a very limited way, something of what they are doing. Boys usually enter into conversation with them, and so the germ of knowledge grows. They are quite surprised to find that land surveying and leveling which seemed so remote and difficult are in reality comparatively simple and comprehensible applications of the same mathematics they have been learning.

This brings me to another phase of applied mathematics, one too often overlooked. So much is said about the exactness of mathematics that few students appreciate sufficiently the fact that mathematics in its applications is always approximate—never exact. The compound interest formula is exact, but its applications are approximate. Architect's plans, engineer's blueprints, even the instruments with which they work are only approximately correct. When you buy a farm or a city lot you get only approximately the amount you pay for, the less you pay

for the land the rougher the approximation. Mathematics is an exact science, not because it does not vary in its applications, but because the higher its development the more nearly exact is its application.

The study of history aids to one's joy in living, to one's sense of social adjustment, because it throws light on the social economic and political problems of present day life and enables one to understand historical references in literature and life. The study of a foreign language is a pleasure, not only because it gives one an added power of communication, but also because it throws light on the life, customs, and characteristics of people of other lands, and opens up a new field of literature. The study of psychology increases one's sense of power because it enables one better to understand the mental and emotional life of others as well as of himself. And, so the mathematics of the high school should do this for the child; it should add to his joy in living, to his sense of social adjustment—not because it has made a mathematician out of him but because it has enabled him to know that its applications are many and varied; that mathematical discoveries are basicly essential to scientific progress; that the difference between the highly civilized world of today and the uncivilized world of the savage is the difference between being mathematical and non-mathematical; that the ethics of mathematics is not its truthfulness but its continual search after truth; that algebra and geometry are not a system of cut and dried rules, theorems and demonstrations, but are universal principles operating the same in every land and among all peoples; that the subject has been thousands of years in developing and is today wider in its scope and applications than ever before.

All this can be effectively done if the teacher will but take advantage of the many opportunities to make the subject fascinating on its own account, and let the applications naturally follow. As Longfellow causes his schoolmaster to say, "There is something divine in the science of numbers. Like God, it holds the sea in the hollow of its hand. It measures the earth, it weighs the stars, it illuminates the universe. It is law, it is order, it is beauty—and yet we imagine—that is, most of us do—that its highest end and culminating point is bookkeeping by double entry. It is our way of teaching that makes it so prosaic."

THE USE OF THE FUNCTION CONCEPT IN FIRST YEAR ALGEBRA[1]

ELEANOR E. BOOHER

We have all tried at various times in our lives to solve riddles and we have all observed others try to solve them too. Now when it comes to solving riddles folks may be classified roughly under two heads: those who give up if the answer does not come to their minds immediately and those who will try for hours. There are the easy givers up and those with the inquiring mind. The very fact that we have devoted a good part of our time to the study and teaching of mathematics must indicate that in our younger days, at least, we were not always with the easy givers up when occasionally a mathematical riddle was propounded to us from back of the teacher's desk. Of course, in these days, mathematics is taught quite differently and many of the problems of the riddle type have given place to a more rational kind of exercise, but even so some of us have had ample opportunity to observe how folks react when riddles are assigned. They may show themselves to be the easy givers up or they may show the inquiring attitude of mind.

Since the report was made by the National Committee on the Reorganization of Mathematics in Secondary Education, much publicity has been given to the use of the function concept as an organizing principle. Just how we can make use of this idea is a question which presents itself as something of a riddle to the members of our profession and so far as I know the answer is not yet forthcoming. But the national committee has succeeded in directing the attention of many folk who do possess the inquiring mind to the solution of this problem. If there is any satisfactory answer it will be found and if not, that fact too will be discovered. I believe that some time in the near future there is going to be an adequate answer, perhaps by one of you, in the form of a text book wherein the subject matter is so organized that the development of the habit of functional thinking is the main aim.

[1] Read Before the Mathematics Section of the High School Conference at University of Illinois, Nov. 19, 1925.

Since Leibnitz and Newton invented the calculus, the function concept has dominated mathematics and all exact science, but it is only within the last decade than anyone seems to have had even a remote idea that teachers of secondary mathematics could make much use of the notion. Although the development of this idea stands out as one of the highest and most significant achievements of man, that does not mean that it is so difficult that no possible use can be made of it in first year algebra. True, indeed, until the seventeenth century, the mathematicians groped about blindly to find a principle of such a fundamental nature that it could be used to unify all of their knowledge. But now that they have bequeathed it to us, we are beginning to wonder if it is not far reaching enough in its significance to simplify and clarify what would otherwise be a somewhat unrelated mass of meaningless processes and manipulations even in first year algebra.

Consider for a moment the role played by the theory of evolution in natural science. It is, I believe, a similar example of a unifying principle by means of which the phenomena of natural science are organized and explained even in the very elementary phases of the subject. It is a mode of thought not too difficult in its simpler aspects for even the young student to grasp. Since it is the method of thinking used by the natural scientist, we start to interpret the phenomena of natural science in the light of it. Because this theory was developed late in the history of man, it does not follow that it is fundamentally more difficult to understand than earlier notions concerning the subject. It is not necessary to know all about it to use it very effectively in interpreting many facts satisfactorily and in relating them all about a central core, and thus to simplify enormously the explanation of many phenomena. The function concept may become to mathematics what the theory of evolution is to natural science—a beautiful example of crude and blundering methods being pushed aside to be replaced by a method much more powerful and much more far reaching in its field of usefulness.

The National Committee on the Reorganization of Mathematics recommends: "The one great idea which is best adapted to unify the course is that of the functional relation. The primary and underlying principle of the course should be the idea of relationships between variables." But how are we to make use

of this recommendation? This small detail is left entirely to us. In the amount of time allowed for this paper only a very limited treatment of this subject will be possible, so I would like to make some suggestions for the application of the function idea to the solution of verbal problems. It seems to me that the verbal problems should hold the most important place in the course and that symbolic manipulation should be a secondary part of the work. The pupil must understand the four fundamental operations with integers and fractions, a little of factoring—in short he must acquire various skills in handling symbols. But for what purpose? Chiefly that he may have the tools which make it possible for him to handle the equations which he sets up in the solutions of verbal problems. It is this type of problem which gives him an opportunity to think about quantities in their relationships to one another and to make some practical applications of the processes which he has learned. If more stress is placed upon verbal problems, there will be less time for the purely mechanical aspects of algebra, but even so it seems to me that there is much to gain. Some of us still spend time teaching complicated problems in complex fractions, long problems in multiplication, division, extraction of roots of polynomials, for all of which there is little or no chance for application in elementary algebra. The demands of the equation are rather simple but the pupil must be thoroughly familiar with all the symbolic manipulations which he will need to use in solving his equations. But if we spend more time on the verbal problem and teach some of this mechanical work in connection with the equations derived, the pupil will see its use and he will have more enthusiasm for the drill work which is necessary for its mastery. It is through the verbal problem, I believe, that the child is likely to get the most from algebra that is truly educative. It is here that his mind has the best opportunity to take hold of quantitative relations and to understand the "why" of the things that he does rather than just the "how" to follow blindly certain rules.

When it comes to the solution of verbal problems under our present method of teaching them, the pupil is likely to come to the conclusion that they are not very important since we have only a few of them now and then. Furthermore, he is likely to take the attitude that they are solved mainly by hit and miss or haphazard methods which consist largely of guessing and

trying first one thing and then another; that there is no general method of analysis which applies to all such problems. If through a consideration of the idea of relationships some fairly general method of analysis could be found, it seems to me that this would be a great aid in developing the ability of our pupils to do orderly thinking. Perhaps we may be able to find a better way to relate and connect the work with verbal problems so that the pupil will feel more keenly that his skill at solving such problems is increasing with each problem solved.

I find that very few teachers who have written any thing at all on the use of the function concept have attempted to give any definite suggestions for its application. They prefer to play safe and make their remarks very general. However, Mr. Paul Ligda of the McClymonds High School, Oakland, California, has just written a book on the Teaching of Elementary Algebra which suggests such a plan. I found this quite interesting, so I shall review it briefly for you. Most of the verbal problems with which we deal in first year algebra are governed by a general formula $R = SxT$ which has also the other two aspects:

$$S = \frac{R}{T} \text{ or } T = \frac{R}{S}.$$

A few examples of these are as follows:

$D = RxT$ (uniform rate formula)
$S = AxB$ (surface area of rectangle, parallelogram, square)
$I = RxP$ (interest formula)
$C = NxP$ (cost formula)

Also each of these problems deals with either one or two situations, and each of these situations is governed by a general or characteristic formula connecting the unlike quantities involved.

Here is an illustration: "A man has 22 minutes to go to the station a distance of 2 miles. If he takes a car which travels at the rate of 1 mile in 8 minutes, at what distance from the station can he get out and walk, if he walks at the rate of 1 mile in 16 minutes?"

This problem contains two situations: the riding situation and the walking situation. The quantities entering each of these situations are time, rate and distance and they are governed by the uniform rate formula $D = RxT$. This formula gives the relationship among unlike quantities in each situation which is not given in the problem but it is a simple relationship governed by a general formula which the pupil knows. The problem contains statements which connect like quantities in the two situations.

Let us now exhibit the relationships existing among the quantities involved in this problem by means of a schematic problem. Let $D = RxT$ represent the quantities entering the walking situation and $d =$ rt (small letters) represent the quantities entering the riding situation. The implied relations connecting like quantities in the two situations are:

(1) $D + d = 2$ (The total distance which the man walks and rides is 2 miles)

(2) $T + t = 22$ min. (The total time spent in riding and walking is 22 minutes.)

(3) The problem gives explicitly the values $R = \frac{1}{16}$ mile per minute; $r = \frac{1}{8}$ mile per minute.

Write vertically the characteristic formula governing each situation. Then summarize all of the relationships between quantities thus:

Walking		Riding
D	$+$	$d = 2$
"		"
$\frac{1}{16} = R$		$r = \frac{1}{8}$
x		x
T	$+$	$t = 22$

Read horizontally this diagram shows the way in which like quantities in the two situations are related to each other. Read vertically it shows how unlike quantities in each of the two situations are related to each other. The problem is now translated into symbols. If any relationship were missing the diagram would automatically indicate that. Now that every quantity is either completely described or related we are ready for the sym-

bolic solution by means of substitution and elimination. To do this eliminate R and r by substituting their numerical values.

$$D/T = 1/16 \qquad d/t = 1/8$$
$$16D = T \qquad 8d = t$$
$$16D + 8d = 22$$
$$D + d = 2$$

Solve these two simultaneous linear equations for D and d.

Translate the result back into words and prove the solution is correct.

As here illustrated, most of the verbal problems solved in first year algebra are problems that contain two statements and thus can be expressed by means of two equations. The method of solution used by practical mathematicians in solving a problem is that of writing down the possible equations and substituting. Why then not introduce simultaneous equations early in the course and let the student obtain a thorough training in substitution, and evaluation; the methods used in work with formulae? This use of simultaneous equations is an essential feature of the method of solving verbal problems by the relationship method. In order to analyze the problem we need as many symbols as there are quantities involved. As long as we adhere to the method of expressing all the quantities in terms of one quantity, preferably the smallest, any general method for analyzing problems is not possible. But if we use a symbol for each quantity and set forth the relationships by means of the diagram, then we can proceed in a systematic manner to eliminate quantities as a part of our symbolic solution.

This type of analysis and solution will apply to most of the problems not in first year algebra. Let us summarize the steps:

I. Analysis (based on function concept, the relationship among quantities).

1. Division of the problem into situations.
2. Recognition of characteristic formula.
3. Write characteristic formula vertically.
4. Division of problem into statements.
   a. Completely describing a quantity.
   b. Explicitly stating a relation between like quantities in two situations.
   c. Recognition of implied statements.

II. Translation into symbols.

III. Symbolic Solution. (Based on equation law.)

IV. Translating Result back into words.

V. Check.

If instead of the usual academic verbal problems, we restate the problems giving them an industrial setting, the practical minded student will take more interest in them. The problem previously used in the illustration may be reworded as follows: "A contractor has 22 days to finish a contract to remove 2,000 tons of rock. If he rents a large steam shovel which can handle 1,000 tons in 8 days, how long should he keep it if his own shovel can handle 1,000 tons in 16 days?"

That this is exactly the same problem as the one which he have previously solved is seen by a glance at the relationship diagram.

Big Shovel		Little Shovel
A	$+$	$a = 2$
"		"
$1/8 = R$		$r = 1/16$
x		x
T	$+$	$t = 22$

The result is the same only the numbers have a different meaning.

Perhaps this method of analysis which is suggested and further elaborated by Mr. Ligda is sufficiently general to make it a valuable aid in the teaching of verbal problems. If it does train the pupil to be constantly watching for relationships among quantities, it is surely worthy of consideration. I believe that we are making a practical application of the function concept in first year algebra when our pupils take that attitude toward verbal problems.

A further application of this method to some problems chosen from a certain well-known first year algebra may prove interesting.

"Two automobiles started at the same time from the same point in opposite directions. The first traveled 5 miles more per hour than the second. At the end of 8 hours they were 360 miles apart. At what rate did each travel?"

Analysis: There are two situations:
Automobile A traveling one direction,
Automobile B traveling the opposite direction.

The characteristic formula governing both situations is uniform rate formula $d = rt$ since the related quantities involved are distance, rate, time.

Use capitals $D = R, T$ for situation A, and small letters $d = r_x t$ for situation B. Write these characteristic formulas vertically.

A		B
D	$+$	$d = 360$
"		"
R	$=$	$r + 5$
x		x
T	$=$	$t = 8$

Divide the problem into statements.

1. Sum of the distance traveled by the two autos is 360 miles. Then $D + d = 360$.
2. The rate of the first auto equals the rate of the second increased by 5 miles. Then $R = r + 5$.
3. Both autos traveled the same length of time—8 hours. Translate into symbols and locate the translated symbols in the diagram.

Symbolic solution: Eliminate T and t by substituting their values $8R = D$ $8r = d$.

In $D \quad d = 360$, substitute $8R$ for D and $8r$ for d and get

$$8R + 8r = 360$$
$$R = r + 5$$

Solve these two equations simultaneously for R and r. $r = 20$ $R = r + 5 = 25$.

Translate the result into words.

Check.

Second example: A boy, Albert, weighing 85 pounds, sits 7 feet from the fulcrum and balances a boy, John, who is sitting 6 feet from the fulcrum on the other side. What is the weight of the boy, John?

Analysis: There are two situations in this problem, Albert sitting on one side of the fulcrum and John sitting on the other.

The characteristic formula governing both situations is $L = WxD$. (The law of the lever.)

Use capitals $L = WxD$ for the situation created by Albert and $1 = wxd$ for the situation created by John. Write these characteristic formulas vertically.

Albert		John
L	$=$	1
"		"
$85 = W$		w
x		x
$7 = D$		$d = 6$

Divide the problem into statements.

Albert weighs 85 pounds.
Albert sits 7 feet from the fulcrum.
John sits 6 feet from the fulcrum.
$L = 1$ (law of leverage).

Translate into symbols and locate these symbols in the diagram.

Symbolic Solution:

$L = 1$
$WxD = wxd.$

Eliminate W, D, and d by substituting their values.

$85.7 = Wx6$
$w = 99\frac{1}{6}$ pounds.

Translate the result into words. Check.

If we are to make a practical application of the use of the function concept in first year algebra, we must aim constantly to develop in our pupils the power to understand the relationships that exist among quantities. A proper emphasis upon the study of verbal problems seems to offer the best possibility for

realizing this aim. It is here that the pupil has presented the kind of situations which develop the "why" and the "what for" and the "what happens if" attitude toward the quantitative side of life. Here he cultivates the habit of expecting to understand rather than just the ability to use rules after he has been shown "how." Some points of view which I believe are important in helping us to make the best use of the verbal problem for developing this type of quantitative thinking are:

First: There must be no teaching of functional theory as such. The teacher must constantly bear in mind that the aim of the course is to develop skill in functional thinking, but in working toward this end, it will not be necessary even to mention the word "function." I am well aware of the fact that there is much disagreement on this point. Many of my co-workers believe that since the function concept is so fundamental to an understanding of mathematics in later courses, we should begin to use the word "function" even in first year algebra; to introduce even then the notion that one variable is a function of another variable. They claim that since a thorough appreciation of this notion is so necessary, it must be presented many times before the student is able to grasp it, and that it is advisable to teach some functional theory even in first year algebra. But since the word "function" has a much different meaning in common parlance than it has in mathematics, I believe that it is better to use such expressions as: "the value of y depends upon the value of x" rather than "y is a function of x." For example, it is better to say, "distance depends upon rate and time" rather than "distance is a function of rate and time." A very informal treatment of the function theory seems to me to be all that is either necessary or desirable in this first year of algebra.

Second: An early introduction of the study of the graph and tables which show relationships between numbers which change together is advisable. This is a type of work which the children enjoy—but that is not all, they work at it from an understanding point of view—just the attitude that we want them to get toward every phase of algebra. It may be noted here that not all of the work with graphs and tables which we usually do is functional in nature. Statistical graphs could not be considered as of much value in developing the idea of a continuous variation among

variables, but such graphs as those showing the relationship between principal and interest, or cost and number are examples of the kind that do develop this idea.

The study of the graph and table makes it possible very soon to introduce the formula and simultaneous linear equations. This early introduction of simultaneous equations seems advisable if verbal problems are to be treated from the viewpoint of the relationships involved among quantities. Most problems contain two situations, so can be more easily expressed and solved by two equations than by only one, thus avoiding the necessity of expressing all of the unknown quantities in terms of one. Children occasionally discover almost unaided, the method of solving problems by simultaneous equations some time before the middle of the second semester when the subject is usually introduced. Just recently, I was trying to teach some verbal problems which are supposed to lead to fractional equations involving one unknown. After school, a boy whom I shall call John, came to me about a problem which he said he had not yet been able to solve. Here is the problem: "Separate 45 into two parts such that $5/9$ of the greater exceeds $1/2$ of the less by 6." John said: "There are two numbers to be found here, are there not? Why can't you use two different letters—one to represent each? If you add them you get 45. If you add 6 to half of the small one you get $5/9$ of the greater one. Isn't there any way to find both these numbers this way?"

It occurred to me that John was about as near ready to learn something about simultaneous linear equations as he would ever be. What should I have done with him?

Third: After the equation has been obtained it must be solved without juggling or artificial devices. The pupil should be able to justify every step with good reasons and the word "transpose" when applied to an equation should not be in his vocabulary. It is an enemy to an understanding attitude of mind. Let him look upon his equation as a question which asks him to find the value of some unknown number. He may write it thus until he is thoroughly familiar with its interrogative nature.

$$5x + 8 \overset{?}{=} 24 - 3x$$

For what value of x is this statement true? There is but one equation law which he need consider. "Whatever is done to one member of this equality must also be done to the other." Let him assume that he has two equal numbers $5x + 8$ and $24 - 3x$. If he wants to know what $5x$ equals he may find out by subtracting 8 from the left members, but in order to preserve the equality or balance, he must also subtract 8 from the right members. This gives $5x = 16 - 3x$.

In the right member he now has $3x$ less than 16. If $3x$ be added to the right member the result will be 16 and clearly now the right member contains no x term. If then $3x$ be added to both members of this equation, the balance will not be destroyed.

$$8x = 16.$$

Divide both members of this equation by 8, to find the value of $1x$.

It may appear to be more laborious to think of all of these additions, subtractions, multiplications, divisions and the like in all such cases, but when the pupil handles the equation this way he understands what he is doing. He is not just juggling symbols.

Fourth: We must ever bear in mind that the main purpose of algebra is to develop the powers of reason. The habit of approaching a problem from the viewpoint of studying the relationships among the quantities involved is a type of thinking which is directly useful to the pupil whether he goes to college or not. The ability to think in terms of relationships among quantities, space and variables is often required of all intelligent human beings—hence there is ever a field in real life into which this type of training can "transfer." In the affairs of real life, good citizens are constantly called upon to estimate, to compare or to see clearly the relations between variable quantities even when exact computations are not made. Hence it appears that the final computation in connections with equations is not so valuable a training as the thinking required to formulate the equation, graph or table from the situations set forth in the problem. A proper use of the verbal problem provides excellent training in just the type of functional thinking which the pupil is sure to find very valuable.

If then we wish to make use of this idea of studying mathematics from the viewpoint of the function concept it is necessary that our work be so organized that the analysis of verbal problems hold the most important place in the course. There should be no teaching of functional theory as such. The graph, the table and the formula are an inseperable trio for expressing relationships among quantities which should be introduced early. Simultaneous linear equations should be introduced as soon as their use makes it possible to solve problems more readily. Symbolic manipulation can be taught as we go along, chiefly as the pupil needs it to help him in the solution of the equations which he sets up. It must be possible for a pupil to give a satisfactory reason for every step which he takes in solving an equation—there must be no juggling. The pupil must develop his ability to see relationships among quantities for this is the very essence of intelligence.

SO LET ME WORK

A conversation inspired by a poem in *The Mathematics Teacher,* Jan. 1925

The preacher speaks:

So let me preach from week to week
That ardent crowds that hear me speak
Remember all my looks and ways,
Though they forget their God to praise.

The tailor speaks:

So let me sew from day to day,
On clothes for work or clothes for play,
That, though they lack both fit and style,
My patrons bless me all the while.

The baker speaks:

So let me cook from meal to meal
That hungry folks who with me deal
Forget all indigestion's ache,
Remembering me, not what I bake.

The telephone operator speaks:

So let me work throughout the week
Connecting men that wish to speak,
That they forget each wrong connection,
But keep me in their recollection.

The taxi-driver speaks:

So let me drive on high or low
That all the fares who with me go
Forget each overcharge and skid,
Remembering me, not what I did.

The teacher speaks again:

So let me teach from day to day
That all I teach fades straight away,
And old grads mutter, each to each
"A kind old soul—but couldn't teach."

HELEN A. MERRILL,
Wellesley, Mass.

A WORD TO THE FOOLISH

If you'd beware of ways erratic
Cultivate the eye quadratic.
Collect your x^2's into *a*,
Corral your x's into *b*.
And everything else is c, c, c
The constant c.
And never anything but c.

Chorus:
ax^2, ax^2, no more;
bx, bx, bx, I've said before;
And everything else is c, c, c, the constant c.
And never anything but c.

Equations ho-mo-ge-ne-ous
Never, never make a fuss.
If vy method makes you squirm
Eliminate the constant term.

MARGARET FEZANDIE,
St. Agatha's School,
New York City.

FIRST AID IN ALGEBRAIC FRACTIONS

If you'd wend your way
Through the forest of fractions,
Pay infinite heed
To these little exactions.

Your work will be always
Both weak and infirm
Unless you distinguish
'Twixt factor and term.

The values of fractions
You'll smash into splinters
Unless you know factors
Are merely "gasinters."

In the dead of the night
You'll wriggle and squirm
If, in reducing,
You cancel a term.

Since Denominators
Are merely a name
Don't add or subtract
Unless they're the same.

And since they are names
And you didn't choose 'em
They remain to the end
Beware, don't lose 'em.

(Although to be sure
For your own information
You cause them to vanish
In an equation.)

With the sign of the minus
In front of a fraction
The whole numerator
Shrieks for subtraction.

A fraction complex
Though fearsome to view
Is just plain division
That's what you must do.

So don't you get scared
In the forest of fractions
Just pay careful heed
To foregoing exactions.

MARGARET FEZANDIE,
St. Agatha's School.

THE CHANGEABLE CHANGELESS NAUGHT

1.

"O, Naughty Naught!" a maiden cried,
"You have me most distracted.
For teacher said you're valueless
When added or subtracted.

2.

"For naught plus naught is nothing,
And naught less naught's the same.
But we must be most careful
To give you your right name.

3.

"It matters not how many times
You may be multiplied,
You can not grow one tiny bit,
In spite of how you've tried.

4.

"And then again when we divide,
You still remain a naught.
Tho' there are very many folks
Who always call you 'ought'.

5.

"But then one day I nearly cried;
I was so aggravated,
I tried to throw you right away;
Oh, dear! It's complicated.

6.

"I had an awful long old sum
In hard old mul'plication;
I thought I'd make it shorter
By your eradication.

7.

"But teacher said, 'You can't do that,
Those naughts are most important;
They multiply those figures there
By hundred, ten, or thousand.'

8.

"And then there was another time,
In doing long division,
When in the quotient I obtained
For you I made no provision.

9.

"The teacher shook her head and said,
'My child, do you not comprehend
That quotient times divisor
Will not give back that dividend?'

10.

"And so it's hard to understand
It certainly is queer,
Why, if you are so worthless,
We must make you appear."

Transition

11.

And then the little naught replied,
"I don't deserve that rating,
To those who know and use me well,
I'm most accommodating.

Main theme

12.

"When by myself I'm valueless,
That's true, without a question;
But when I'm not alone my worth
Depends on my position.

13.

"You may add me or subtract me,
Multiply me or divide me,
But you can not change my value
When no figure is beside me.

14.

"Sometimes on me the digits nine
Depend for their valuation;
I also help them to define
The terms of their relation.

15.

"When at the right of a decimal point,
You may keep me or discard me.
But when there is no decimal there,
You must treat me with the utmost care.

16.

"And sometimes you may wish to find
The rate percent of discount,
You may add on naughts, one, two, or more,
If you're sure to put a decimal point before.

17.

"And thus, my child, I could cite many cases
Where I am a friend most true,
And when you get to know me well
You'll find me a friend to you."

Irene G. Hagan,
Wm. O'Rogers School,
New Orleans, La.

EFFECT OF CERTAIN TYPES OF SPEED DRILLS IN ARITHMETIC

A. I. MESSICK
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A. The Problem

The purpose of this study is to determine whether it is advisable to emphasize speed or accuracy in teaching addition to pupils in the fourth and fifth grades in the elementary school.

B. The Method

Two hundred seventy-two pupils in the fourth and fifth grades of the Bloomington, Indiana, public schools took part in this experiment. The fourth and fifth grades in the McCalla and Central schools were used. For convenience in drill work the two drill groups were divided into four sections each, according to their grade divisions in school. Each section had its own room and teacher. The sections for each grade were A and B; A being the advanced section of each grade. Half of the speed group came from McCalla and half from Central; the accuracy group was also divided in this manner. The following table shows the number of pupils in each section, the school to which each belonged and the drill process which was used. From now on the schools will be referred to as school M. and school C. Thus 4.A.M. means grade 4 A from the McCalla school.

TABLE I

School	Grade	Drill Process	No. Pupils Drilled
M.	4 A	Speed	33
M.	5 B	Speed	34
C.	4 B	Speed	32
C.	5 A	Speed	37
	Total		136
C.	4 A	Accuracy	33
M.	4 B	Accuracy	32
M.	5 A	Accuracy	40
C.	5 B	Accuracy	31
	Total		136

(Note) M.--McCalla School
C.--Central School

Each grade was drilled by the writer four minutes each day for twenty days. Each group had its particular method of practice emphasized each day, Addition was selected because it is generally considered to be the most important of the four fundmentals since it serves as a basis for the other three.

For the daily practice and drill eight problems were used. Each problem consisted of three columns of nine digits each. The problems for each day were different so far as answers were concerned, but were of the same degree of difficulty. Those problems involved the forty-five combinations and were compiled from the Courtis Practice Pads and the Courtis Arithmetic Tests, Series B, forms one, two, and three. Following is a sample of one day's problems.

(PROBLEMS FOR ONE DAY)

------------Day. Name---------------------------- Grade------------

537	267	584	877	326	648	474	258
685	854	157	845	770	396	787	885
542	124	617	981	753	389	591	100
904	358	624	693	199	374	106	874
511	938	467	184	469	839	869	226
988	333	151	772	643	919	451	355
559	493	245	698	423	336	336	142
127	775	233	245	698	423	336	336
323	239	233	238	173	342	533	351

These problems were used during each of the twenty days except during the first three days when they were broken up into problems of one and two columns each. This was done as an aid to the pupils in getting started in the drill work.

At the beginning, before the drill work for the twenty days was begun, the Courtis Tests, Series B, form I, in addition were given in order to get a starting point for each group. Immediately after the expiration of the twenty days of drill the Courtis Arithmetic Tests, Series B, form II, was given in order to find out the degree of improvement in each group. The Instructions as printed at the top of the Courtis Tests, Series B, forms I and II, were followed in the giving of these tests.

During the first few days special care was taken to explain to each group the importance of the particular drill process by which it was being drilled. The speed group was made to see

the necessity of adding rapidly and getting its problems right. The accuracy group likewise was impressed with the importance of adding slowly and being certain that each problem attempted was added correctly. It was also made clear to each group that if they added too rapidly or too slowly it ceased to be adding and would do them harm. It was also explained to each group what the purpose was in drilling them for twenty days, i.e. to see how much they could improve in addition.

The following daily instructions were given during the drill period. The speed-group.—"Write your name and grade on this paper and turn it face over on your desk. Today we have eight problems and you will be given four minutes in which to work as many of them as you can. Add rapidly and think carefully as you work. We have found that as a rule the best adders are the ones who work most rapidly. Should you finish before the time is up check your problems by adding in the opposite direction: i.e., if you first added them from the bottom to the top then re-add from top to bottom and place your second answer immediately under your first answer. When I say 'pencils up' raise your pencils in the air. When I say 'go' begin adding. When I say 'stop' raise your pencils in the air at once."

Instructions to the accuracy group.—"Write your name and grade on this paper and turn it face over on your desk. Today we have eight problems and you will be given four minutes in which to work as many of them as you can. Add slowly and think carefully as you work. Be certain that each problem that you attempt is added correctly. Should you finish before the time is up check your problems by adding in the opposite direction: i.e., if you first added from the bottom to the top then re-add by going from the top to the bottom and write your second answer immediately under your first answer. When I say 'pencils up' raise your pencils in the air. When I say 'go' begin adding. When I say 'stop' raise your pencils in the air at once."

Giving the problems of this study occupied the time between February 20, 1923, and March 28, 1923. A record of the absences was kept for each group during the entire drill period. The classes in school C were given their drill each day beginning at 1:05 P. M. The classes in school M. were given their drill each day beginning at 2:40 P. M. From ten to fifteen minutes

were given in each of the eight sections each drill day in distributing papers, giving instructions, giving the drill, and collecting the papers. The results of each day's practice were given to the pupils on the following day so that each pupil knew just how he or she was getting along. The main purpose of this procedure was to stimulate and motivate the work of the pupils from day to day and to relieve any monotony that might tend to arise. The result was that the interest in all classes was exceedingly high during the entire drill period.

At the close of the drill period only those pupils who were present and took the Courtis Tests the first time were permitted to take the last set of the Courtis Tests. All tabulations which appear in the following chapter were done by the writer.

Table II shows the results of the twenty days' drill by grades. Each grade is composed of two sections, one from school C., and one from school M. The results of the two fourth grades follow: The group drilled for accuracy made a gain of 34.6 per cent in speed while the group drilled for speed made a gain of 43.6 per cent. This shows that the speed group improved 9 per cent more in speed than did the accuracy group. In accuracy the speed group improved 49.1 per cent, while the accuracy group improved 93.1 per cent. This shows that the group drilled for accuracy made a gain of 44 per cent more than did the speed group.

For the purpose of a closer analysis of the results obtained by the different drill methods, an examination of the per cent of pupils which gained, lost and remained the same in speed and accuracy is necessary. In speed the group drilled for speed 1.5 per cent higher score than did the accuracy group. 6.2 per cent more pupils in the speed group lost than lost in the accuracy group. 7.7 per cent more of the pupils remained the same in the accuracy group than lost in the speed group. In accuracy the group drilled for accuracy had 1.7 per cent more pupils to gain than did the group drilled for speed. .6 per cent more of the pupils drilled for accuracy lost than did the speed group. 2.3 per cent more pupils remained the same in the speed group than did in the accuracy group.

TABLE II

Drill Process	Section	Grade	Per Cent of Examples gained in Speed	Acc.	Speed Per Cent of Pupils Gained	Lost	Same	Accuracy Per Cent of Pupils Gained	Lost	Same	No. of Pupils Drilled
Speed	4 A.—M. 4 B.—C.	4	43.6	49.1	89.2	7.7	3.1	68.9	20	11.1	65
Acc.	4 A.—C. 4 B.—M.	4	34.6	93.1	87.7	1.5	10.8	70.6	20.6	8.8	65
Speed	5 A.—C. 5 B.—M.	5	23.2	20.8	78.9	11.	10.1	60.7	28.5	10.8	71
Acc.	5 A.—M. 5 B.—C.	5	37.5	41.9	89.8	2.5	7.7	64.7	20.1	15.2	71

(Note). M...McCalla School
C.....Central School
Acc.....Accuracy

The data for the two fifth grades is as follows. Table three indicates that the group drilled for accuracy made a gain of 14.3 per cent more in speed than did the speed group. In accuracy the group drilled for accuracy made a gain of 21.1 per cent more than did the speed group. In speed 10.9 per cent more of the pupils drilled for accuracy made a gain than did those drilled for speed. 8.5 per cent more of the pupils drilled for speed lost than did those drilled for accuracy. 2.4 per cent more of the pupils drilled for speed remained the same than did those drilled for accuracy. In accuracy 4 per cent more of the pupils drilled for accuracy made a gain than did those drilled for speed. 8.4 per cent more of the pupils drilled for speed lost than did those drilled for accuracy. 4.4 per cent more of the pupils drilled for accuracy remained the same than remained the same for the speed group.

Table III gives the data for the entire accuracy group and for the entire speed group. This table is compiled from Table II.

TABLE III

Drill Process	No. of Pupils Drilled	Per Cent of Examples gained in Speed	Acc	Speed Per Cent of Pupils Gained	Lost	Same	Accuracy Per Cent of Pupils Gained	Lost	Same
Accuracy	136	36	67.5	88.7	2.	9.3	67.7	20.3	12
Speed	136	33.4	35.	84.	9.4	6.6	64.8	24.2	11

Table IV indicates that the accuracy group made a gain of 36 per cent in speed while the speed group made a gain of 33.4 per cent. This shows that the accuracy group made a higher score

by 2.6 per cent. In accuracy the group drilled for accuracy made a gain of 67.5 per cent, while the group drilled for speed made a gain of 35 per cent. This shows that the accuracy group made a higher score by 32.5 per cent.

In speed 4.7 per cent more of the pupils drilled for accuracy gained than did those drilled for speed. 7.4 per cent more of the pupils drilled for speed lost than did those drilled for accuracy. 2.7 per cent more of the pupils drilled for accuracy remained the same than did those drilled for speed.

In accuracy 2.9 per cent more of the pupils in the accuracy group gained than did those in the speed group. 3.9 per cent more of the pupils in the speed group lost than did those in the accuracy group. 1 per cent more of the pupils drilled for accuracy remained the same than did those drilled for speed.

Conclusions

Since:

1. Table IV indicates that the accuracy group exceeded the speed group by 2.6 per cent in speed and by 32.5 per cent in accuracy.

2. From the same table it is seen that 4.7 per cent more pupils gained in speed when accuracy was emphasized than when speed was emphasized. Again 7.4 per cent more of the pupils drilled for speed lost in speed than did those drilled for accuracy. Also 2.7 per cent more of the pupils drilled for accuracy remained the same in speed than did those drilled for speed. Again in *accuracy* the group drilled for accuracy had 2.9 per cent more pupils gain than did the speed group; they were 3.9 per cent more pupils of the speed group lost than did for the accuracy group; and the accuracy group had 1 per cent more pupils to remain the same than did the speed group.

3. Accuracy is more important than speed.

Therefore:

1. From the viewpoint of *speed* it makes little difference which is emphasized speed or accuracy.

2. From the viewpoint of *accuracy* it is much better to emphasize accuracy.

3. In teaching addition to pupils of the fourth and fifth grades of the elementary schools it is better to emphasize accuracy rather than speed.

THE ACCURACY OF THE VALUES FOR PI

SIDNEY WALCK
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Many values have been used for Pi, ranging from 3 to the value out to 707 decimal places. At present nearly every school boy knows it as 3.1416, or about 22/7. The error using 3 is 4.51%, for 3.1416 it is .00234%, or an error of one inch on the circumference of a circle having a radius of 0.93 mile. Thus we see that the value to a few places is sufficiently accurate for most practical purposes, and as far as a geometrical construction is concerned, take the following example: inscribe in a given circle a square, and to three times the diameter add one-fifth of the side of the square; assume that the result is the circumference. The error on the circumference of a circle having a radius of 5 inches is .00172 inch, much less than the actual error of construction with ordinary drawing instruments.

There have been three epochs in the consideration of the quadrature of the circle or finding the ratio of Pi. The first, extending from the earliest times to about 1650 A. D., is chacterized by innumerable attempts at finding Pi by purely geometric methods.

The second period extended from the invention of calculus to the middle of the 18th century. In this period the methods of analysis replaced the geometric methods, infinite series and products being used to approximate the value of Pi.

The third period, from the middle of the eighteenth century to the present time, is characterized by the efforts not so much to discover values of Pi as to find the nature of this number, whether or not it is rational and whether or not it is transcendental.

The most accurate value of Pi that we have is the value worked out to 707 decimal places by William Shanks in 1873. Let us calculate the circumference of a circle having a radius of one million light years from its radius, using the value of Pi to 707 places, assuming that there is an error of one point in the

707th place. We would not be able to see the error with our best microscope. This statement, however, is too conservative to indicate the real accuracy of the value. The electron is the smallest division of matter that we know, having a radius of about 4×10^{-14} inches. The radius of an electron would be 5.29×10^{669} times as large as the error, while the circumference of the large circle would be only 5.94×10^{37} times the radius of an electron. If the error were laid off on the circumference of an electron it would subtend at the center an angle of 3.90×10^{-665} seconds. To bring the magnitude of the error within our conception let us multiply it by $(5.94 \times 10.^{37})^{18.0847}$. Then the error would amount to about one inch. To get a good idea of the increase produced by multiplying by 5.94×10^{37} once let alone to the 18th power, let us apply the process to the inch. The value obtained would extend around the earth at the equator 37,600,000,000,000,000,000,000,000,000 times.

IMPROVEMENT IN ALGEBRA TEACHING

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In an article by Harry C. Barber in the October, 1925, number of The Mathematics Teacher there are some very interesting questions asked of the teacher of mathematics about what he is going to do for a ninth grade girl if she comes to his classes for the pupose of studying Algebra. Now I am the teacher and I believe I can answer some of those questions quite satisfactorily without waiting for this fourteen-year-old daughter to become a grandmother, for I feel very certain that if we wait for the final answer to be given by experimenters and psychologists we will allow many generations of fourteen-year-old sons and daughters to become grandparents and yet the answer will not be given to the satisfaction of all.

To begin with, let me say that since your daughter, Mary, I shall call her, is not studying Algebra for any vocational purpose at this time, all the questions you ask about her algebra you may ask with exactly the same force of her teacher in English, in Latin, in Science and in History and the answer will be as definite in one case as in any other. Each is a great branch of human knowledge and if one fails to make some acquaintance with any one of them he misses something of value in the development of his life and mind. If he neglects to become somewhat acquainted with three or four of them he will come short of what we are accustomed to call a person of culture and education.

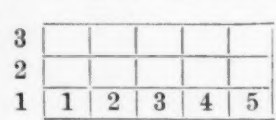
To be specific, if Mary is a girl of fair average intelligence she will learn more about the nature of numbers than she learned in Arithmetic. She will find that problems which in Arithmetic were quite difficult, in Algebra are comparatively easy. She will find that in order to do this she must exercise a little patience and learn how to use algebraic language before she can hope to understand completely the meaning and significance of Algebra and to her astonishment she will find that the answer which so staggered her father, namely, "Algebra is where you use letters," is really quite as simple as to say: "M stands for Mary, J stands for John and *a* stands for apple." This is probably about all I can say that will be of any interest to Mary at the

present time and it is all that is necessary. This answer will be quite satisfactory if Mary has been interested in Arithmetic. On the other hand if she has a preconceived idea obtained from her young friends or her parents that Algebra is dry and useless, no answer or argument I can produce will be satisfactory. If Mary has been fond of History then her history teacher in High School can very quickly tell her the gain in studying History, but if she "just hates history" no person on earth can convince her that she should study History. The same may be said in the case of any other branch of learning.

I wish now to raise a question about one point in Mr. Barber's discussion. Of course understanding and thought on the part of the pupil are what we are after and new ideas should be introduced as they are needed, and my question is simply this. Are we as logical as we think we are when we introduce Algebra with the formula? In presenting a formula are we leading the pupil to think for himself or are we teaching him to go through certain motions because we tell him that will get the right result?

I know of the work of the National Committee and have spent three years or more in earnestly trying to make Algebra more real and reasonable for beginners and after two years of earnestly trying to introduce it with formulas I have been persuaded by results to call a halt and admit that we, in our school, are not succeeding any better than previously.

Is the formula a thought producer or can it be used in solving a thought prbolem? Of all the devices which have been discovered in Algebra for solving problems the formula is the one *par excellence* for obtaining a result with the minimum of labor of either mind or muscle. Take, for example, the simple little formula for the area of a rectangle $A = lw$. When the first semester student is told that the length is 5 inches and the width is 3 inches and he substitutes in the formula $A = 5 \cdot 3$ hence $A = 15$ square inches, he has found the correct result, but has he done any constructive thinking? I believe he has not. If he is to find the area of a rectangle let him draw a figure and reason thus: In the first row of squares there are five square inches. Since there are three such rows there are therefore three times five square inches or fifteen



square inches. Let this be repeated every time the area of a rectangle is to be found and, after much practice, it will not seem cumbersome and the pupil will understand what he is talking about. The same thing holds true for practically every formula used in our Algebra. If what we want is to have the pupil take no step in Algebra without understanding the meaning of what he is doing and to have him solve no problem without reasoning it out for himself, (and I admit freely I do want that) then why introduce the subject with formulas which are a cut-and-dried ready-for-use machine which turns out answers with but little understanding of the problems solved? It seems to me that the building up of many simple equations which we do not call formulas at all is a much more valuable thought exercise than solving formulas.

Perhaps these statements have been made a little extreme, but candidly are they any more extreme than to insist that "transpose" is such a bad word that it must never be used in polite society? I agree that this word should not be used when we first solve an equation, but when the pupil has repeatedly added the same number to, or subtracted the same number from both sides of the equation he sees for himself that an equal number with the opposite sign always appears on the "other side" of the equation and he will use transposition in spite of the teacher and I want him to do so. Then why not use some word to denote the process instead of the cumbersome phrase "adding the same number to both sides of the equation"? If *transpose* is not a good word then why not invent a better one and use it instead of using a whole long sentence? If it is a sin to allow a pupil to "transpose" without explaining every step of the process throughout all the first year's work, then it is a much greater sin to allow him to use formulas which he probably never understood very well and certainly does not attempt to explain with every application.

TEACHING PLANE GEOMETRY WITHOUT A TEXT-BOOK

THEODORE STRONG
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In teaching a class in Plane Geometry every teacher has most probably encountered each of the following difficulties:

(1) That of introducing the class to the subject in such a way as to give its members enthusiasm as well as a good foundation;

(2) That of adapting the text-book to his individual taste, for no text-book, however excellent, will completely satisfy any teacher as to the sequence of theorems or as to the proofs employed;

(3) That of bringing the class to the point of attacking original exercises in a fashion truly efficient. It is with the idea of suggesting a partial solution to these three problems that this paper is being written.

The problem of introducing the pupils to Plane Geometry in satisfactory fashion is probably as difficult as any of the three mentioned. And it is made the more difficult by the fact that to the experienced teacher the introductory material is so familiar. It is many years since the teacher himself leaped the chasm that separates algebra from geometry, and he is prone to forget both the breadth of that chasm and how different the geometric part of the mathematical world seemed when compared with the algebraic. In beginning a course in Plane Geometry with the aid of a good text-book, the teacher is tempted to assign the class a certain number of pages in the introductory chapter, then on the following day to discuss these and assign the next few pages. The introduction is thus completed in perhaps a week or a little more than a week, and the class then plunges into the study of formal proofs. Now the class has understood the new material, has accepted the definitions and admitted, in general, the truth of axioms and postulates; but that is about all. This material—the foundation stones of the geometric edifice—is not in reality theirs to use. Their conceptions of essential elements of geometric figures are still vague. They cannot define these terms in accurate and concise language unless

they have memorized and can repeat them parrot-fashion; and this is not knowledge. The axioms and postulates are still new and strange tools with whose use they are quite unfamiliar. And the result of this is that many pupils, especially those for whom the subject at best would be quite difficult, begin to memorize the proofs of theorems with all the unhappy consequences that ensue.

Can the teacher avoid this unfortunate condition? The writer firmly believes so. And the solution of the problem is this:—give from three to four weeks to the introduction, even in a course that devotes only one year to the subject. For this work the class, in the hands of an experienced teacher, needs no text-book and is better off without one. Let the teacher challenge the class to develop its own definitions. The pupils will not be able to do this unaided, but they will surely become dissatisfied with vague conceptions, will develop an appreciation for accurate and compact statements; and once a definition reaches its final form, the entire class will not only be able to use the term appropriately but to define it properly. For the past two years the writer has done this with very happy results. He made his own introductory chapter, following rather closely an excellent text-book and including additional material that seemed worthwhile. After two revisions the main topics in their order are as follows:

(1) Comparison of arithmetic and algebra and suggestions as to what geometry was like and how it compared and contrasted with arithmetic and algebra.

(2) A brief study of logic bringing out clearly the nature of conditional statements, showing that all statements are in a real sense conditional, and leading to the conclusion that both the converse and negative of a true conditional statement are not necessarily true and must always be proved.

(3) Definitions of terms bringing out the fact that some terms were so simple that they could not be well defined.

(4) Need for demonstrative geometry brought out by proving the unreliability of intuition and so leading to the introduction of axioms and postulates, and concluding with the proof of the equality of vertical angles as a specimen and initial theorem.

In all this work the class was held responsible only for what had been committed to their note-books, and nothing was written in these until all understood it and were satisfied with it, if a

definition, or convinced of its truth, if an axiom or postulate. Written recitations were given daily and marked most strictly; home-work was assigned that gave ample opportunity to become familiar with the use of the straight-edge and compass; and, in order that the pupils might familiarize themselves with some of the history of geometry, each was required to write a paper on Euclid. As was stated before, over three weeks were consumed in this work, but it paid large dividends. The class had gradually become thoroughly at home in their new and very different subject; their old conceptions had been clarified and they were willingly learning to express themselves much more accurately and concisely; every tool that they needed for further work was theirs in a very real sense; and the subject was not so difficult as fascinating.

When the class engaged in the study of Book I, every theorem, save the more difficult, was introduced as an original exercise. In practically every case the right method of attack was discovered by some member of the class. To be sure, it was generally one of the best pupils who made the discovery, but often, by proper suggestion a weaker pupil could be led to the same happy enlightenment. In all the oral work there was no grading of the pupils, and this proved a source of great encouragement to those for whom the subject was more difficult. The grading was done on written recitations given almost daily and consuming from ten to twenty minutes each. In consequence, the teacher had, in the remainder of the forty-five minute period, ample opportunity in the general class work to help these weaker pupils to develop initiative and systematic methods of attack.

But there were difficulties, and difficulties that the writer believes need not exist. Proofs in the text-book were often abbreviated in order to preserve the unit page, and it was necessary to show this plainly to the class. A more serious difficulty lay in the teacher's preference for a proof different from that in the text or for a different sequence of theorems, not because of any alleged fault in the text-book but because of the teacher's individual taste. A third difficulty arose from the fact that pupils occasionally anticipated the class-room discussion by studying the advance theorem, thus completely destroying its value as an original exercise.

To remove these inconveniences the writer proposes, during the coming year, to teach entirely without a text-book. A sequence of theorems has been determined upon, a sequence that includes every theorem and construction given in the syllabi of the College Entrance Examination Board and suggested by the National Committee on Mathematical Requirements in its report entitled "The Reorganization of Mathematics in Secondary Education." Many theorems and constructions heretofore included in text-books but not necessary for the proof of those on these syllabi are made original exercises except those that are too difficult for such use. It is firmly believed that a course so organized, a course in which practically every theorem and construction is presented as an original exercise, a course in which the pupil does not study a text-book but makes and studies his own, will solve in large measure the several problems just mentioned. Proofs will be as detailed or as abbreviated as seems necessary, regardless of the unit page; the order of development will be that best adapted to the teacher's personal preference; every new theorem can be treated as an original exercise as much or as little as desired; and, in addition, the pupil will find his material for study not in text-book and note-book but in the latter alone.

Finally, we have to consider the problem of teaching the pupils how to attack original exercises efficiently. The writer believes that in treating the theorems as original exercises, the class will unconsciously be learning exactly how to proceed in the case of the exercises. Throughout the year they will become habituated in an effective method, and of course they will from the beginning of Book I be applying this method to original exercises.

Two additional details are worthy of note. In order to keep the class, especially the weaker pupils, familiar with the theorems and their proofs, especially the more important, the class is always held responsible for the proof of every theorem, construction, or corollary quoted in the proof of the new theorem; and proofs of these may be called for in the written recitations. This provides a continuous review throughout the year and increases the pupils' familiarity with those geometric truths that are most useful in the original exercises.

Individual differences are provided for by a list of original exercises more difficult than those taken up in the class. This is placed on the bulletin board and pupils whose grade is sufficiently high may attempt them, working on them independently of each other and of the teacher. Additional credit is given for this work, failure not being penalized in the least. This not only stimulates the leaders of the class but also the rank and file, for they are tempted to improve their work on the essentials in order to join the privileged group whose extra efforts cannot lower their grades but in practically all cases do raise them.

Plainly, such a course is out of the question in the case of the teacher who has several classes in the subject, or classes that are very large; for it would be impossible to correct all the papers that would be written. And, in any case, the expenditure of much time and effort on the part of the teacher is an essential prerequisite to success. The proofs of many theorems will have to be mimeographed to avoid the waste of time that would be involved in writing the proof as developed or in dictation of the proof afterward. Original exercises and practical applications will have to be selected and inserted at proper places throughout the five books. These will not be easy tasks, but is there not good reason to believe that the results will justify the efforts? If a reasonable degree of success has attended the effort to teach theorems as originals when a text-book has been used, it is a fair prediction to say that there should surely be a real improvement in a course especially planned to provide the maximum opportunity for such work. And success with original exercises is the "summum bonum" in Plane Geometry: therefore any procedure that contributes to this goal is well worth whatever it costs, provided the price can be paid.

NEWS NOTES

A program of the twenty-third annual meeting of the Association of Teachers of Mathematics of New England was held in Boston, December 5, 1925. The program included:

(1) "Informal Approach to Geometry," by Lucy E. Allen of the Thayer Academy;

(2) "Some Class Room Problems of the Mathematics Teacher," by W. F. Downey of the English High School;

(3) "The Use of Statistics in Astronomy," by Doctor W. J. Luyten of the Harvard Observatory;

(4) "Mathematics and Culture," by Professor A. N. Whitehead of Harvard University.

The members of the Council of the Association are: Professor Lennie P. Copeland, F. E. Lane, Harry D. Gaylord, Harold B. Garland, Harry C. Barber, Olive A. Kee, Professor George D. Birkhoff, Annie W. Doughty, Professor Eleanor C. Doak, and Rolland R. Smith.

The fifty-seventh annual meeting of the Association of Teachers of Mathematics of the Middle States and Maryland was held at Columbia University, December 1, 1925.

Mr. H. E. Webb of the Central High School, Newark, read a paper on "Elementary Geometry and its Foundations." Mr. H. Eugene Seymour, New York Supervisor of Mathematics, spoke on "Suggestions for the Improvement of Instruction in Secondary Mathematics Based upon Observations in New York State."

The officers of the Association for the coming year are: Dr. J. Ross Smith, President; Mr. Vevia Blair, Vice-President; Mr. Clarence Scoborio, Treasurer, and Miss Elsie Bull, Secretary.

Professor José Arteaga, who is in charge of instruction in secondary mathematics in all secondary schools in Mexico, is studying the curriculum and methods of teaching Mathematics in secondary schools in the United States, preparatory to reorganizing courses in secondary mathematics throughout the schools of Mexico.

Edwin W. Schreiber, head of the Department of Mathematics, Proviso Township High School, Maywood, Ill., was recently elected a Fellow of the American Association for the Advancement of Science.

The twenty-third regular meeting of the Association of Mathematics Teachers of New Jersey was held at Atlantic City, Oct. 12, 1925.

The program included: 1. A Critical Study of the Value of the Topics in High School Mathematics, by Ernest H. Koch, Jr., Dean of the Brooklyn Technical High School; 2. Symmetry in Elementary Geometry, by Andrew S. Hegeman, Central High School, Newark; and 3. Limitations in Geometrical Constructions, by Professor Emory P. Starke, Rutgers College.

The officers of the Society are: President Arthur W. Belcher, East Side High School, Newark; Vice-President, J. P, Stout, High School, Lakewood, and Secretary-Treasurer, Andrew S. Hegeman, Central High School, Newark, N. J.

DISCUSSION

The undersigned has read with great interest former articles by Evans, a recent article by Nunn, and an article in the January issue by Webb on the status of demonstrative geometry teaching in the secondary school. The status *is* unsatisfactory. The difficult thing to understand is the apathy of intelligent teachers towards this question.

Doubtless the reason for the present chaotic condition is our failure (with all due respect for official reports), to designate the primary aim in the *teaching* of demonstrative geometry. A syllabus of propositions does not answer this question.

I hope that one outcome of Evan's paper at Washington will be the formation of a committee to study this question.

Since we are concerned primarily with the question of teaching geometry in a secondary school, such a committee should contain enough secondary school teachers of geometry to present ade-

quately the present status of such teaching. And, unless we can get a report of sufficiently wide adoption so that its provisions will appeal to examining bodies, little real effect may be obtained here in the East.

H. H. Hart,
Montclair, N. J.

I hope that Prof. Reeve's "Objectives in Teaching Mathematics" in the November Mathematics Teacher gets the attention and discussion it merits. It is quite clear that ninth-grade algebra must be considerably reconstructed if it is to compete in educational value with the new programs in Latin, modern foreign languages, English, social studies, and science.

Prof. Reeve's objectives as formulated give us too much to do I am sure. (This was probably his intention with a view to better final formulation.) We cannot do so much if we are to make a course which will deserve to become a part of the instruction of all ninth-grade children, and if we are to avoid that curse of first-year algebra—hurry.

Most of us will agree that factoring has been much overdone. Also, a limited number of formulas, well considered, is better than too many. Several of the broader generalizations listed are not in keeping with the way a ninth-grade child thinks. The trigonometry unit surely ought not to be omitted; it shows the child something of the power of mathematics. Quadratics ought to be included if only because solving them by factoring is for the child one of the best applications of factoring, and solving them by a formula is one of the best applications of a formula.

Harry C. Barber,
Charlestown High School,
Charlestown, Mass.

NEW BOOKS

An Arithmetic for Teachers—William F. Roantree and Mary S. Taylor; the Macmillan Company, 1925.

The authors have succeeded in steering a safe and sane course between the extremes of an academic course in arithmetic and a miscellaneous collection of methods and devices. Each chapter is divided into two sections, the first of which is devoted to the teacher's knowledge, and the second to the methodology of the subject. In the first section of the chapter, sufficient historical material is presented to give the teacher a working knowledge of the development of mathematical concepts and processes. The reliable historical items are in pleasing contrast to the somewhat bizarre material which has recently appeared in works of this character. No subject in the elementary school curriculum is more intimately bound up with its evolution than is arithmetic.

The sections on methodology give abundant evidence of sound pedagogy, wrought out in the stress and strain of preparing teachers to teach arithmetic to children. Sufficient objective development is indicated to give significance both to concepts and to processes. The chapter on problem solving will be welcomed by many teachers. The chapters on denominate numbers, measurement of geometric figures, percentage and its applications are well written and in keeping with present day methods. A chapter on development and drill will repay the reading by any teacher. It is a pleasure to record the fact that testing, valuable as it is, is allotted its due space, twenty-five pages in a total of six hundred ten.

This book will be a valued addition not only to the library of the teacher in service, but as a text in training classes and reading circles.

L. LELAND LOCKE.

January 8, 1926.

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(To be continued)

IMPORTANT ANNOUNCEMENT

In accordance with a motion passed at the last meeting of the National Council of Mathematics in Cincinnati, the First Yearbook is now ready for distribution.

Table of Contents

1. A General Survey of the Progress of Mathematics in our High Schools in the Last Twenty-Five Years—Professor David Eugene Smith, Teachers College, Columbia University.
2. On the Foundations of Mathematics—Professor Eliakim Moore, University of Chicago.
3. Suggestions for the Solution of an Important Problem That Has Arisen in the Last Quarter of a Century—Professor Raleigh Schorling, The University of Michigan.
4. Improving Tests in Mathematics—Professor W. D. Reeve, Teachers College, Columbia University.
5. Some Recent Investigations in Arithmetic—Professor Frank Clapp, University of Wisconsin.
6. Mathematics of the Junior High School—Mr. William Betz, Rochester, New York.
7. Mathematics and the Public—Professor H. E. Slaught, The University of Chicago.
8. Some Recreational Values Secured in our Secondary Schools Through Mathematics Clubs—Miss Marie Gugle and others, The Columbus Schools, Columbus, Ohio.
9. Mathematics Books Published for Secondary Schools and for Teachers of Mathematics in Recent Years—Edwin W. Schreiber, Proviso Township High School, Maywood, Illinois.

The price is $1.10 prepaid. The number of copies is necessarily limited, so all persons desiring the book should get their orders in early.

All correspondence relating to the book should be addressed to C. M. Austin, High School, Oak Park, Illinois.

THE MATHEMATICS TEACHER

VOLUME XIX MARCH, 1926 NUMBER 3

A REVIEW OF PROFESSOR MOORE'S PRESIDENTIAL ADDRESS[1]

HARRY ENGLISH
Chief Examiner, Public Schools, Washington, D. C.

(Presidential Address Reprinted in the Yearbook)

On December 29, 1902, Professor E. H. Moore of the University of Chicago delivered the Presidential Address before the American Mathematical Society, "On the Foundations of Mathematics." It is with great hesitation that I make any attempt to consider this epoch-making pronouncement of the views and visions, ideals and hopes, of this noted mathematician and general thinker as regards mathematical content and methods.

To review Professor Moore's paper as it should be reviewed would require more time than the few minutes which are allotted me, and even if there were ample time, I am quite sure that my best efforts would fall far short of even adequate justice to its far-reaching influence. For this reason and because of the present day conditions in our mathematical world I should prefer to have you regard this paper as merely a stimulus to re-read Professor Moore's address with the utmost care, filling in for yourselves those thoughts which will naturally arise because of your own peculiar environment and point of view.

Merely for the sake of clarity I shall divide my remarks into the following general headings:

A. Preliminary Statements.
B. The General Settings in the Years 1901, 1902, 1903.
C. Desirable Changes Advocated by Professor Moore.
D. Goals Set Up That Have Not Been Achieved, OR
Proposals as to Which There is Still Controversy.

Necessarily there will be difference of opinion as to the amount of emphasis to be placed on the word "desirable." Necessarily also questions will at once arise as to the possibility and practicability of desirable changes, the rapidity with which they can be made and the various modes and methods of making them.

[1] Read at the meeting of the National Council of Teachers of Mathematics, Feb. 20, 1926, Washington, D. C.

A. Preliminary Statements

The first part of Professor Moore's paper concerns itself with a brief general discussion of abstract mathematics on the one hand and pure and applied mathematics on the other. Keynotes are pointed out which are afterwards applied to elementary mathematics as follows:

1. Abstract mathematics involves the use of a definite system of undefined symbols in conjunction with a definite system of postulates. Its symbolism gives power, rigor and simplicity of procedure but, of course, does not obviate essential difficulties in a scientific, logical analysis.

2. All science, logic and mathematics included, is not fixed by metes and bounds, but in ideals as well as in achievements is a function of the epoch.

3. The division into deductive and inductive sciences cannot be wholly maintained and is understood in its entirety by the devotee of science only when he has reached a considerable degree of maturity.

4. The attitude of the practical mathematician towards the abstract mathematics and especially its symbolism, is akin to that of the practical physicist toward theoretical mathematics, the practical engineer toward theoretical mathematics, the practical engineer toward theoretical physics of mathematics.

5. Pure mathematics in its ordinary acceptation may be taken as the systematic development of the properties of the fundamental mathematical notions of number and form which were among the earliest to enter into the formulas of science, these formulas being more or less exact descriptions of phenomena.

6. Arithmetic and geometry closely united in mensuration and trigonometry early reached a high degree of advancement, but the development of the generalized literal notation of algebra largely in response to the demands of mechanics, astronomy and physics in the 17th century bound arithmetic and geometry still more closely together into analytic geometry and calculus, those mighty instruments of research which were developed further and further in the 18th and 19th centuries.

7. Especially during the 19th century came the critical reorganization and development of the foundations of pure mathematics (i.e., 'the mathematics of precision') independent of ap-

plied mathematics (i.e., 'the mathematics of approximation'), causing a gap towards the bridging of which agencies are at work.

Such is the gist of the preliminary statements made by Professor Moore on December 29, 1902.

B. The General Settings in the Years 1901, 1902, 1903

1. John Perry in 1901 at the first session in Glasgow of the Section on Education of the British Association in connection with the section on Mathematics and Physics delivered a paper on "The Teaching of Mathematics" emphasizing strongly the necessity for emancipation from Euclid as the sole authority in geometry especially with reference to sequence of propositions, and laying emphasis on practical mathematics based upon the notions underlying the infinitesimal calculus taken as axioms, map drawing to a scale, the use of squared paper, the use of decimals as numbers, etc.

He laid down as an absolute pre-requisite that the boy should be made familiar by every possible way (experiment, illustration, measurement, etc.) with the ideas to which he applies his logic and should be thoroughly interested in the subject studied.

2. This effort of Perry was part of a movement already in existence in England for thirty years designed to relieve secondary teachers from the burden of a too precise examination system imposed by the great examining bodies, and was followed in England by the appointment of a strong committee under the chairmanship of Professor Forsyth to report upon improvement in the teaching of mathematics, especially elementary mathematics, and means of accomplishment. This report was distinctly favorable, though in a conservative way, and initial changes were started.

3. In the United States the interest in the Perry movement was widespread and intense, and the years 1901, 1902, 1903 were notable because of the almost spontaneous formation of associations of teachers of mathematics for the vital improvement of mathematical teaching, and though necessarily they differed as to content, methods of approach, points of emphasis, sequence of topics, etc., all were animated by an earnest desire for the true development of the flesh and blood child with due regard for his personal equation.

Among the associations active in the consideration of the Perry Movement may be mentioned: The New England Association of Teachers of Mathematics; The Central Association of Science and Mathematics Teachers, with headquarters in Chicago, which was enthusiastic for the Perry Movement; The Association of Teachers of Mathematics in the Middle States and Maryland (including the District of Columbia) which was more conservative.

The Association of Teachers of Mathematics in the Middle States and Maryland was in the process of formation during the year 1903 and had its first meeting in New York City, November 28, 1903. At that meeting the present writer read the first paper presented to the Association on "The Laboratory Method of Teaching Mathematics," setting forth the laboratory method which had been in use in the schools of Washington, D. C., for many years, the suggestive plan of the Central Association of Science and Mathematics Teachers, and the general conditions obtaining at that time. Questions that are vital in 1926 were vital in 1903 and many desirable changes now advocated were then advocated by many.

This brings us to the consideration of that part of Professor Moore's address dealing with changes, bearing in mind:

(1) That Professor Moore spoke from the standpoint of the pure mathematician following very closely many of the ideas advanced by Professor Perry.

(2) That the International Commission on the Teaching of Mathematics after a comprehensive and exhaustive investigation as to facts and conclusions, issued a series of reports in 1911, the most vital and interesting of which from our immediate point of view being those of the three American Commissioners, Professor Smith, Professor Osgood, and Professor Young.

(3) That the World War of 1914 while interrupting the work of the Commission created new fields of investigations, and methods of dealing with them which has profoundly affected all educational problems especially those pertaining to mathematics.

(4) That the Junior High Movement was not in the vision of Professor Moore.

C. Desirable Changes Advocated by Professor Moore

1. Diminished emphasis on the systematic and formal side, and increased emphasis on the practical side (e.g., on arithmetic computation, elements of mechanical drawing, graphic methods, etc.) by connecting abstract mathematics with subjects which are naturally of interest to the boy and by using [1]those theorectical processes that may be checked by laboratory processes, so as to give to the very young students in their impressionable years in a thoroughly concrete and captivating form, in addition to trigonometry, the wonderful new notions of the 17th century, the essentials of analytics and calculus, and the practical use of the first mathematical tools in such a way that they will be interested in the theory of the tools themselves and so regard mathematics as a fundamental reality of the domain of thought and not as a collection of symbols and arbitrary rules.

2. It is pedagogically sound at first to take a large body of basal principles, reserving for later years a more philosophical treatment starting with a smaller number of such propositions.

3. As regards elementary schools: The kindergarten methods should be extended into the grades, but not to such an extent as to prevent appreciation of the abstract methods that come later; and the pupils should be trained in powers of observation, experimentation, reflection and deduction by direct connection with matters of a thoroughly concrete character (e.g., drawing, paper folding, elementary mechanical drawing, construction of models, graphical representations of one magnitude depending on another and their concrete meanings, use of co-ordinate paper) thus enriching and vitalizing the materials and methods, leading to the study of intuitive geometry and linking geometry with numerical and literal arithmetic.

4. As regards secondary schools: There should be no watertight compartments. The systematic development of theoretical physics and of theoretical mathematics should come later. For the benefit of the student, at first the physics should be practical and likewise in geometry at first there should be emphasis of its concrete side with exercises in informal deduction followed by the more formal deductive geometry, omitting many axioms and taking for granted ordinary properties of measurement and

[1] It is taken that Professor Moore does not mean "only those theoretical processes."

motion, early leading up to co-ordinate geometry and *calculus in a concrete way (*calculus is debatable) without taking into account those wonderful related discoveries in abstract geometry which should be reserved for later college and university years. It would be undesirable to attempt to make complete reforms in arithmetic, algebra and geometry all at once.

5. The laboratory method based on a two-period plan preferably, as a means of reform:

(a) This reform calls for a thorough laboratory system or method of instruction, handling pupils individually or in groups so as to develop in each pupil a true spirit of research and appreciation of the practical and theoretical methods of science, the graph and the number and measure relations mutually aiding each other.

(b) Though it may be admitted that it is impossible to concentrate on more than one thing at a time, yet the problem, given as a matter of course, should involve the use of instruments, but should not be given specifically to show their use. Also in teaching one thing relations should be illuminated by the use of others, and the smaller elements of mathematical routine should be attached to and used in laboratory problems.

(c) All results should be checked by at least two distinct methods so as to render the student independent of authority, while in practical problems the amount of error permitted in the final result should be indicated.

(d) The formal proof of a theorem should be preceded by concrete computational or graphical or experimental means so as to awaken appreciation of the importance of the theorem and a desire for formal proof.

(e) Each student studies things and not words, utilizes the experiences of others in close co-operation with the instructor and all are mutually benefited.

(f) The students in the secondary schools will see the vital relation between the fundamental relations of trigonometry, analytic geometry and calculus, when they are closely connected with concrete phenomena and practical education, and with such momentum, in college will be enabled to develop individual and effective interests even in formal abstract mathematics.

(g) Such a full secondary mathematical course in laboratory methods in mathematics and physics (again this unification is a question) is best for all who expect to specialize in college whether in pure mathematics, or pure physics, or in mathematical physics, or in astronomy or in any branch of engineering, but this is to be done gradually, by evolution not revolution, by voluntary close co-operation of teachers and students, by use of computational and graphic methods, colored inks and chalks and by laying emphasis on the comprehension of propositions.

(h) Teaching should be made a profession and the junior colleges by means of the laboratory method can offer training courses in the pedagogy of secondary mathematics in a wonderfully effective way, courses which the college graduates sadly lack, however well trained they may be in content work.

6. Secondary schools are somewhat hampered by specific college requirements and there should be conferences so as to lead to close co-operation.

7. A national organization of science teachers should be formed.

8. The American Mathematical Society should give continuous expert attention to the matter and should form two divisions: (1) Research; (2) Pedagogy.

9. All organizations should take up the matter and form a clearing house for all ideas.

D. Goals Set Up That Have Not Been Realized Or Proposals as to Which There Is Still Controversy

Before stating these, attention is here called to the fact that in Professor Moore's address (and in nearly all of the literature at that time and for some years afterwards and indeed in late years) reference is always made to the boy and not to the girl.

1. That the movement for the enlargement of strong secondary schools by the addition of two years of junior college and the last two or three years of the grades to form an institution like the German gymnasium or the French Lycée will afford a fine opportunity to carry out these reforms is doubtful or at least debatable.

2. That colleges, especially junior colleges should proceed independently to reform their elementary college courses without waiting for the secondary schools to reform their courses and that the secondary schools should likewise proceed independently to reform their courses without reference to the elementary schools is doubtful or at least debatable for many reasons. It is acknowledged that much material will have to be transferred to the lower schools. This with changes in methods would cause an unpleasant hiatus.

3. That the secondary schools should proceed to organize algebra, geometry and physics into a thoroughly coherent four years' course (a suggestion made several times) comparable in strength and closeness of structure with the four years' course in Latin to be given by men who have received expert training in mathematics, physics and engineering is doubtful, or at least debatable. It is to be here noticed that the readjustments in the Latin courses have been very great since Professor Moore delivered his address.

4. That under the laboratory system outlined there should be a thorough laboratory system or method of instruction in unified mathematics and physics, and whether there can be a double period for such laboratory work is doubtful or at least debatable.

5. That the fundamental problem is the unification of pure and applied mathematics by so arranging the curriculum that no branching shall be recognized is doubtful or at least debatable.

6. That there shall be a continuous relation of mathematical problems with problems of physics, chemistry and engineering is doubtful or at least debatable.

7. That the average boy will understand the ideas advanced under the laboratory method which illustrate the uniformity of convergence is doubtful or at least debatable.

The above is an earnest attempt, faulty though it is, to give some idea of Professor Moore's remarkable address delivered nearly a quarter of a century ago. The more one studies it in the light of present day conditions, the more one is impressed with its prophetic character.

The views expressed are not put down in a dogmatic form. In many cases they are opinions based upon the then known

facts and conditions or are in the form of inquiries; in many cases they are expressed as component parts of a vision; in many cases views expressed in one part of the address must be read in connection with views on the same subject advanced under another part of the address to obtain their true significance.

Since then slowly but surely many of the barren spots of mathematics have been removed and the whole subject has been made more vital. The foundations for effective co-operative work were laid in 1902 and 1903; the objectives to be reached, which had long been advocated by so many, though not with the same emphasis have been more or less harmonized, though, of course, all advocates cannot come to the same conclusion.

The World War served to show that common understandings, mutual respect of those holding divergent views, mutual sacrifice for the common good, and kindred attitudes were necessary and desirable not only for war, but for peace activities, not only in trade but in education. The necessary reorganization of conditions necessitated readjustment, readaptation to life's activities and preparation to meet them, while the methods used during the war necessarily had an effect on all educational, social and governmental activities.

For some time the attitude towards all things has been a critical consideration of, and interpretation of ascertained facts to be modified if necessary as the field is extended and the methods of appraising and measuring the values and relations of those facts are perfected.

The tendencies in mathematics as viewed by progressives have been in existence for many years. Those who were more conservative have resisted too great encroachment. Each side had advocates of unquestioned ability and integrity. So long as there was no opportunity for actual meeting and friendly unofficial discussions, there were long range controversies productive of further misunderstandings. This difficulty is being removed and mutual confidence has been established which marks a red letter day in the progress so much desired by all who have in mind the proper development of the child as a social unit of most worth to himself, to the community, to the state.

This mutual confidence which is gradually expanding has made it possible to employ a scientific method of approach in mathematics akin to that used by scientists and especially by the

medical profession: testing by laboratory methods and other methods of analysis for the properties and characteristics of the product under consideration, and then modifying these results by all the other considerations which go to make up the true estimate that should eventually hold.

This is a long and tedious process for the scientist and especially for a member of the medical profession, while the supreme ability, that of accurate diagnostics is even now in its infancy.

The application of the process to education and especially to mathematics necessarily must be slow, and set backs must be expected through lack of understanding of its purpose, its mode of application and action taken on results. An outstanding contribution during recent years has been the introduction and the development of the scientific method of testing in mathematics for: abilities to be developed; useful knowledge to be acquired; proper material to be obtained; sequence of topics in textbooks linked with and based on environment with gradual proper development to prepare for real life:—in short a scientific method of testing so as to accomplish the objectives which have been so earnestly desired and which heretofore have been based on individual opinions of experts which were controlling and rightly so within their special sphere of influence but which might or might not apply in all localities.

The scientific method in mathematics is now entering upon a second and necessary phase—that of the extension of the tests and the perfection of tests already used based upon the records of actual results properly tabulated and extending over a number of years and scientifically checked with actual results.

The makers of these tests are the last to urge that they are sacrosanct. It is claimed, however, that they are of inestimable value in the hands of the experienced teachers as well as of the inexperienced teacher far away from help. These tests do not constitute a universal panacea to cure all mathematical ills nor are they machines to be used by any one to decide the ultimate fate of a pupil. Such work is for the expert. Much less are they to be used indiscriminately for diagnostic purposes. Such work[1] is in the stage of infancy.

[1] These tests, not in use when Professor Moore delivered his address, have confirmed nearly all of his views in a most impressive manner, while at the same time they afford a very sure basis, as pointed out, for future work.

The outstanding results of the past twenty-five years of progress have been pointed out and need not be restated here. The same spirit of mutual help and confidence in impersonal methods scientifically applied will continue and the next few years will witness a wonderful progress in all matters pertaining to things mathematical, in comparison with which the progress of the past twenty-five years, great though it is, will seem very small.

THE FOURTH DIMENSION AND HYPERSPACE

BY MISS THERESA TROMP
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When I was asked to speak to you twenty minutes on the subject of the Fourth Dimension and Hyperspace, I at first refused, for I thought to talk to mathematicians and scientists about the fourth dimension would be as stale as the news that the earth is round, but after questioning a few of my friends as to what they understood by the fourth dimension and hyperspace, I found that very few of them had a clear idea of what it meant. Not that I know anything about it, for I like to belong to the School of Socrates. When Socrates lived in Athens, it was reported that he was the wisest man in Athens, and Socrates said that he thought they were right, for he said that he knew nothing, and knew that he knew nothing, while the others knew nothing, and thought they knew something. Now I claim to know nothing, and know that I know nothing, but I will attempt to explain to you a few things about what the others think they know.

I will attempt to explain to you the essential character of hyperspace and hypersolids; some of the evidences of a fourth dimension, and a few of the latest developments of the subject; also the utility of speculations on the fourth dimension and hyperspace.

The fourth dimension, as you all know, is a space at right angles to length, breadth, and thickness. Because we are not conscious of fourth dimensional space we must have recourse to analogy to help us understand it. We ascertain the relations between two dimensions and three dimensions and then establish these relations between three dimensions and four dimensions. Just as a plane moving in a direction not contained in itself would generate a three space, so a three space moving in a direction perpendicular to its three dimensions would form a hyperspace. Beginning with a straight line, draw another line perpendicular to it, that will determine the direction of the second dimension. At their point of intersection, draw a third line perpendicular to the first two, that will determine the direction of the third dimension. Now at their intersection, imagine a line drawn perpendicular to those three, that would determine

the direction of the fourth dimension. Obviously, it would contain the point of intersection, but the remainder of the line would stretch out in an unimaginable direction from the inside of that point. The fourth dimension of a solid object lies in a direction within it. A line perpendicular to those four lines would lie in the direction of the fifth dimension and so on.

With the principles of plane and solid geometry, and with making only one assumption, that a line can be drawn perpendicular to three space, we can build up an entire geometry for the fourth dimension. The fourth dimensional geometry is built up by analogy. Just as one reasons what one would do to figures in a plane to form solids, so by analogy one reasons what would be necessary to do to solids to form hypersolids. Just as a square has four corners, and a cube eight, so a hypercube would have sixteen corners. A square is bounded by four lines, and if you move the square through space perpendicular to itself to generate a cube, we find that the cube has the four edges of the square in its first position, the four edges of the square in its final position, and that each corner of the square traced an edge as it moved through space, or twelve edges. By analogy a hypercube formed by a cube moving perpendicular to three-space would have twice the number of edges of the moving cube plus an additional edge for each corner of the cube, or thirty-two edges. The cube is bounded by six planes, or twice the number of planes in the moving square plus a plane for each edge of the square, so the hypercube would have twice the number of planes in the moving cube plus a plane for each edge of the cube, or twenty-four planes. A hypercube would be bounded by eight cubes. One cube at the initial position of the moving cube, one at its final position, and one cube made by each plane of the moving cube as it moved through space. The boundaries of hypersolids would be volumes. The solid of any space becomes the boundary of a corresponding solid in the next higher space. A section cutting a hypersolid into two parts would be three dimensional. A plane cannot separate two parts of a hypersolid any more than a line can separate two parts of a solid. The rules for finding the volume of hypersolids are derived as the corresponding rules for volume are derived in ordinary geometry. Thus the hypervolume of a hypercube would be the volume of the base multiplied by the altitude. Quite an extensive geometry has already been worked

out for the fourth dimension, even including the nature of the hecatonicosahedroids, hypersolids with one hundred faces. These are only a few of the geometrical deductions, and the dramatic results that would accompany a fourth dimension are even more amazing.

By imagining two dimensional beings living in a plane and unable to perceive anything of a third dimension, we get a vivid idea of our own relation to fourth dimensional space, and a consideration of what ought to be their attitude toward any conception of a space of three dimensions, makes clearer what ought to be our attitude toward conceptions of a higher space. Just as in three space a point can go in and out of a square without touching the boundary, so in four space a body could escape from our strongest prison without going through any of the walls. By the fourth dimension you could remove the contents of an egg without breaking the shell. You could take your right glove and flip it over in the fourth dimension and it would come back a left glove. Our densest solids would be open to inspection from the fourth dimension.

Some mathematicians have tried to show that time is the fourth co-ordinate. If time is the fourth dimension, what would one mean by the fifth dimension or the hundredth dimension? This view is too anthropocentric to be plausible. Time is an attribute of all space. Even in a world of one dimension there would be duration or the time it takes an object to move from one position to another. This view arose from the theory that our space is curved in a fourth dimension. We must resort to analogy to understand what this would mean. Imagine a soap bubble with a cone being pushed through it. A being living on the two dimensional surface of the bubble and having no comprehension of a third dimension would see only the successive cross sections of the cone, which he would notice were continually changing, and he would probably call it aging or growth just as we do here. He would have no suspicion that what he is observing is the motion of a solid object in a third dimension. He would see only the two dimensional cross sections of a three dimensional object. Should a four dimensional object come within the field of our vision, we would see only its three dimensional cross section. Einstein claims that our space is curved in a fourth dimension, and that we are being pushed through from a fourth

dimension with a speed equal to that of light, just as the two dimensional surface of a soap bubble is curved in the third dimension and a three dimensional cone can be pushed through it. A three dimensional being could see the whole cone at once, or the whole phenomena occurring which the two dimensional being would interpret as the stream of time. Likewise a four dimensional being could see our whole past, present, and future at once instead of as a stream of time. In any given space, the next higher dimension although spacial has a temporal appearance. Space, time, and motion, are all aspects of relation and some mathematicians claim that they are convertible terms.

The fourth dimension is a perfectly logical concept. The only difficulty comes when we attempt to apply it to our experience and find that it transcends our experience. The hyperdimensionality of space is no longer a question, but the hyperdimensionality of matter is still unproven.

One of the arguments against the fourth dimension is that matter does not move in that direction. Matter may move in the direction of the fourth dimension without our knowing it. Just as the cone being pushed through the bubble would be moving in the direction of the third dimension, but the two dimensional beings living on the surface of the bubble would not know it and would see only the two dimensional cross sections. We cannot assume that because we do not observe a phenomenon it does not exist. We know that there are light waves below the red and above the violet end of the spectrum which are invisible to the eye. There is a possibility that we are a part of four dimensional space with physical limitations which confine us to three dimensional space, and with limitations of our senses which prevent us from perceiving anything outside of this space. Several observations have already been made which seem to indicate that matter also moves in a fourth dimension.

In a plane objects as observed by two dimensional beings would have an infinitesimal thickness in a third dimension, otherwise they would be mere shadows, so in our world objects as we observe them would have an infinitismal thickness in a fourth dimension. It is interesting to note that from the discoveries in recent years which seem to indicate that matter moves also in a fourth dimension, the most evidences of this have been found in the infinitesimal, in cellular and molecular activity.

In our experience we find the phenomena of symmetry only in the minute, such as is produced in plants and animals and in corpuscular and atomic action, and not in the large masses, such as mountains. You all know that symmetrical figures can be made to coincide by giving each a quarter turn toward each other through the next higher dimension. From this point of view, symmetrical figures may be regarded as resulting from a splitting of one figure in a given space and an unfolding in the next lower space. This would explain many of the phenomena of corpuscular action. Also certain chemical isomers appear to differ only in that their molecules are symmetrically instead of identically formed and that in some cases there seems to be free passage from the one form to the other without any manifestation of chemical change. An explanation, though not the only one, would be rotation through the fourth dimension.

Another probable evidence is found in electricity. In a plane the axis of rotation is a point, in three space it is a line, and by analogy in four space rotation would be about a plane as an axis. The electric current forms a closed circuit, which is a geometric image of rotation about a surface. The electric current seems to be a hyper-vortex in the ether revolving about a plane as an axis. The entire nature of electro-magnetic action seems to indicate that the world we know is a cross section of a four dimensional world.

Another interesting possible evidence of the fourth dimension is found in the orbital motion of spheres. Imagine a wire spring suspended in the air and being slowly immersed in a bucket of water. The cross section of the spring where it intersected with the water would be an ellipse which would describe an orbit on the surface of the water each time a coil of the spring was immersed. Now imagine this spring itself being composed of many minor springs of different mass and motion, each intersecting the surface of the water in an ellipse and describing an orbit each time a coil of the spring was immersed. By analogy the cross sections of spiral vortices moving through our ether from a fourth dimension would be spheres moving in orbits, which is exactly what we find in nature. Thus on the hypothesis that the forms of our space are cross sections of four dimensional forms, much of the phenomena of nature would be accounted for, and matter would then be simply vortices in the ether.

From Einstein's discoveries, it seems that our entire universe is a four dimensional hypersphere of finite volume, which by analogy would make it the successive cross sections of one-fifth dimensional helix. For if there is a fourth dimension, there is no reason to stop there. If there is a fourth dimension, it would logically follow that there are any number of dimensions.

Dimension means measurement, and the whole problem of the fourth dimension or any other dimension rests on the nature of space. Space and time have reference to a system of reckoning distances and durations, and one of Einstein's great discoveries is that there are many such systems of reckoning, one as fundamental as the other, and that our frame of space and time is a purely earthly appanage. Einstein's discoveries are the latest development of the subject of space and dimension, and he has arrived at his conclusions by scientific experiments and mathematical calculations. Until recently it was thought that there was an absolute space and an absolute time which could be measured, but we know now that the measurement of space intervals and time intervals are not absolute but relative, and change with the position and motion of the observer. A cubic yard is not an absolute chunk of space. What may measure as a cubic yard to one observer, may measure up as a cubic foot to another in a different position in the universe on a body of different velocity. Time perceptions there would also be different. If our time perceptions were a hundred thousand times faster, events would appear to us a hundred thousand times slower. Anything moving as fast as our express trains would require a delicate experiment to show that it was moving.

Thus it would seem from Einstein's discoveries that our frame of space and time is a purely geocentric view and that space, time, and motion, are not independent entities, but aspects of reason. Einstein calls space and time, "terrestrial adhesions to thought." Kant calls space and time, "synthetic judgments a priori." The majority of philosophers, mathematicians, and physical scientists, agree that space, time, and motion, are not entities, but conceptions of the human mind in its relation to nature. In the last analysis the fourth dimension deals with the fourth step in the reason's comprehension of infinity.

Of what use are speculations on the fourth dimension? Of about as much use as to know whether the earth goes around the

sun or the sun goes around the earth. Space is as properly an object of study as the planets. There is nothing about the fourth dimension that is any more mystical than there is about the other three, and a study of it gives us a clearer idea of the nature of space. It gives us a deeper insight into geometry. Just as a study of solid geometry enlarges our comprehension of plane geometry, so solid geometry is illuminated by a study of hyperspace. It enlarges our mental horizon, and throws light on the nature of our mental equipment. Such speculations sometimes lead to very fruitful results.

USE OF THE INVENTORY TEST IN PLANE GEOMETRY

By LEONARD D. HAERTER
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In the course of the last few years there has been an extensive reorganization of the course of study in mathematics in grades seven, eight, and nine. One phase of this reorganization movement deals with the incorporation of much geometric material. A large amount of simple construction and drawing is being done with the ruler, compasses, and protractor. Through the application of both arithmetic and algebra in the field of geometry the child is becoming familiar with the names and shapes of a large number of plane and solid figures. In addition, many of the simpler notions of geometry are being experimentally obtained by the pupils in these grades.

It was our opinion at the University High School, Minneapolis, Minnesota, that the students beginning the study of plane geometry in the tenth school year must have acquired by means of this revised course of mathematics in grades seven, eight, and nine, many geometric facts which we formerly tried to give them in the first few weeks of the study of plane geometry. We felt that it was most worthwhile to know what information they already had of plane geometry before they began a formal study of that subject.

Accordingly a test was devised covering the beginning of a course in plane geometry. The test, a copy of which appears below, consists of eighty-eight items divided as follows:

A yes-no test of thirty-six items; a completion test of twenty items; a computation test of ten items; and a construction test of twenty-two items.

The following test has been made to determine how much you know about the subject of plane geometry before you actually begin to study it. You are urged to read each question carefully and to answer each one to the best of your ability.

Yes-No

Read the following questions carefully. Write either yes or no before each one.

_______1. Does a triangle have three sides?

_______2. Is the area of a rectangle equal to the base times the height?

_______3. Does a right angle contain 180 degrees?

_______4. Are all right angles equal?

_______5. Are all the angles of a rectangle right angles?

_______6. Does a right triangle contain two right angles?

_______7. Is the area of every parallelogram equal to the product of two adjacent sides?

_______8. Are all the sides of every rectangle equal?

_______9. Is the area of a circle equal to Πd^2?

______10. Is the circumference of a circle equal to $2\ \Pi r$?

______11. Is the area of a triangle equal to the product of one-half the base times the altitude?

______12. Is the sum of the interior angles of a triangle equal to 180 degrees?

______13. Are two lines in the same plane that do not meet perpendicular?

______14. If one line is perpendicular to another are the angles formed acute angles?

______15. Is a quadrilateral a geometric figure of five sides?

______16. Is an acute angle less than a right angle?

______17. Can a triangle have two right angles?

______18. Does a right angle contain 190 degrees?

______19. Does a straight angle contain 180 degrees?

______20. Can the sum of two acute angles equal a straight angle?

______21. Is the sum of two complimentary angles equal to 180 degrees?

______22. If two angles of a triangle are equal are the sides opposite these angles equal?

sides

______23. Is the sum of two angles of a triangle ever less than the third side?

______24. If the sum of two angles is equal to two right angles, are they complementary?

______25. If the opposite sides of a quadrilateral are parallel, is the figure a parallelogram?

______26. Are the acute angles of a right triangle complementary?

______27. Are two triangles always congruent (equal) if three sides of one are equal respectively to three sides of the other?

______28. Can more than one perpendicular be erected at a given point in a line?

______29. Are the opposite sides of a quadrilateral always equal?

______30. Is the area of a rectangle equal to the area of a triangle having the same base and altitude?

______31. Is an obtuse angle larger than 180 degrees?

______32. If a triangle has all its sides equal is it equilateral?

______33. Is the area of a right triangle one-half the product of the legs?

______34. If two angles of one triangle are equal respectively to two angles of another triangle, is the third angle of the first equal to the third angle of the second?

______35. If two sides of a triangle are equal, are the angles opposite equal?

______36. If three angles of one triangle are equal to three angles of another triangle, are the triangles always congruent (equal)?

Completion Exercises

Write one word in each blank to make the sentence state a geometric fact.

1. A line passing through the center of a circle and having its ends in the circumference is called a______________________.
2. An angle containing 90 degrees is called a__________________angle.
3. In a triangle the line drawn from the vertex perpendicular to the opposite side is called an___________________________.
4. A rectangle contains________________________right angles.
5. If one line is perpendicular to another, the angles formed are______ ____________________________angles.
6. A line drawn from the center of a circle to the circumference is called a____________________________.
7. If a rectangle has all its sides equal it is called a______________
8. If all the sides of a triangle are equal, the triangle is called________.
9. If a triangle contains a right angle, the triangle is called a__________ triangle.
10. In a right triangle the side opposite the right angle is called the ____________________________.
11. Two angles whose sum is equal to 180 degrees are called____________.
12. The_____________________of a_____________________is equal to Πr^2.
13. If two sides of a triangle are equal the triangle is called____________.
14. A straight line that touches a circle in only one point is called a ____________________________.
15. A hexagon is a geometric figure of______________________sides.
16. The sum of the interior angles of a triangle is equal to______________ degrees.
17. If two triangles have______________sides of one equal respectively to _______________sides of the other, the triangles are________________.
18. The line joining the opposite vertices of a quadrilateral is called a _____________________.
19. The square of the___________________of a right triangle is equal to the________________of the________________of the other two sides.
20. An equilateral triangle has each angle equal to______________degrees.

Computation Exercises

1. A rectangle has a base of 20 feet and an altitude of 8 feet. What is the area? Ans.____________sq. ft.
2. A triangle has a base of 12 feet and an altitude of 10 feet. What is its area? Ans.____________sq. ft.

3. The sum of two angles of a triangle is equal to 120 degrees. How large is the third angle? Ans.____________
4. One of the acute angles of a right triangle is 50 degrees. How large is the other acute angle? Ans.____________
5. Two parallel lines are cut by a transversal so that one of the angles is equal to 30 degrees. How large is its alternate interior angle? Ans.____________
6. An acute angle contains 70 degrees. How many degrees are there in its complement? Ans.____________
7. The vertex angle of an isosceles triangle contains 20 degrees. How many degrees in each of the base angles of this triangle? Ans.____________
8. A central angle of a circle contains 40 degrees. How many degrees in its intercepted arc? Ans.____________
9. One leg of a right triangle is 6 feet and the other is 8 feet. How long is the hypotenuse? Ans.____________
10. How many degrees are there in the sum of the angles of a quadrilateral? Ans.____________

Construction Exercises

Draw freehand as neatly and as large as possible the figures and lines asked for.

I. The triangle:

1. Draw an isosceles triangle here.
2. Mark the base AB.
3. Draw the altitude to that base.
4. Draw a bisector of one of the angles at the base.
5. Draw a line through the vertex of the triangle parallel to the base.
6. Draw a right triangle here.

II. The quadrilateral:

7. Draw a rectangle here.
8. Draw a diagonal of this rectangle.
9. Draw a square here.
10. Draw a trapezoid here.

III. The circle:

11. Draw a circle here.
12. In this circle draw a radius.
13. In this circle draw a chord.
14. Draw a line tangent to this circle.
15. Draw a central angle of this circle.

IV. Angles.

15. Draw a central angle of this circle.
16. Draw an acute angle here.
17. Mark the angle so that it can be read angle ABC.
18. Draw its complement. Place a (1) in this angle.
19. Draw its supplement. Place a (2) in this angle.

V. Parallels:

20. Draw two parallel lines here.
21. Draw a transversal to these parallels.
22. Mark a pair of alternate interior angles by (1) and (2).

On the opening day of school this test was given to 74 pupils who were beginning the study of plane geometry for the first time. All the pupils had pursued a revised course of study in grades seven and eight, and a course in general mathematics in grade nine. It is interesting to note in the abulations which follow, just how much geometry they knew.

Yes-No Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Times missed	0	5	6	10	7	9	34	5	31	28	7	2	23	15	17	5	8	1

No.	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
Times missed	1	11	33	20	12	32	23	28	16	6	21	5	18	8	30	7	17	47

Completion Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Times missed	22	2	41	6	8	17	12	18	3	27	43	22	38	60	31	3	49	52	51	20

Computation Test

No.	1	2	3	4	5	6	7	8	9	10
Times missed	6	6	8	11	35	34	26	68	36	20

Construction Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Times missed	17	14	22	19	45	17	5	21	12	40	9	13	60	63	56	14	38	45	45	11	13	30

Each item of this test was valued at one point and the score of each pupil was obtained by adding together the total number of correct replies that appeared on the paper.

The median score of the class on the test was 63. A brief comment on the summary will be illuminating.

Yes-No Test

1. Seventeen questions were each missed ten times or less.

2. Twenty-five questions were missed less than twenty times each.

3. Only eight questions, numbers 7, 9, 10, 21, 24, 26, 33, and 36, were missed by more than one-third of the class. An examination of these questions will readily explain the frequency of error. Numbers 9 and 10 would be missed almost as frequently by a class who had already studied plane geometry. Number 7

is a trick question, while numbers 21, 24, and 26 deal with the notion of complementary angles, a notion which is readily confused with that of supplementary angles by the pupils.

4. Only one question, number 36, was missed by more than half the class. We have evidence from the results of the Minneapolis Geometry Test, devised by Miss Anna Thomas of Minneapolis, that this question is almost as frequently missed by pupils who have studied plane geometry for one-half year.

Completion Test

1. Seven questions, numbers 3, 11, 13, 14, 17, 18, and 19, were missed by over half the class. This portion of the test was apparently the most difficult.

2. Nine questions were missed twenty times or less. The wording of these questions undoubtedly gave many pupils trouble; while the strangeness of this type of test was also confusing.

Computation Test

1. Only question 8 was missed by over half the class.

2. Questions 5, 6, 7, 8, and 9 were missed by over one-third of the class.

3. Numbers 1, 2, 3, and 4 were missed by less than one-seventh of the pupils.

Construction Test

1. In the construction test, numbers 5, 10, 13, 14, 15, 17, 18, 19 and 22 offered special difficulty. They deal, however, primarily with things met only after many weeks in plane geometry, and with the matter of complementary and supplementary angles.

2. Half the questions were missed by less than twenty persons in the class.

On the showing made by the pupils on the test we decided to spend no time whatever in reviewing these facts as a part of the regular class work. We believed that undoubtedly a brief review by the pupil out of class would readily recall these things to his mind. Each student was therefore given a slip of paper bearing his name and the numbers of the problems missed, and was told to find the proper answers to the questions missed. References were given and a sufficient number of tests posted on bulle-

tin boards for the pupils to consult. After one week the test was given again. The median of the class was raised from 63 to 80. Only one pupil made a score below 50.

A summary of the results made on the second giving of the test follows.

Yes-No Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Times missed	0	4	3	1	1	3	20	8	15	17	6	3	14	2	9	2	4	0

No.	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
Times missed	0	3	17	0	4	15	12	14	6	1	12	1	7	2	17	8	1	39

Completion Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Times missed	14	3	36	2	3	5	8	1	4	8	13	11	10	22	12	2	18	30	43	4

Computation Test

No.	1	2	3	4	5	6	7	8	9	10
Times missed	0	4	2	6	16	9	8	52	13	8

Construction Test

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Times missed	0	0	6	4	26	4	2	2	3	13	2	9	27	22	18	5	25	16	17	3	3	4

Criticisms of two kinds may be raised.

1. University High School has a very select group of students.

In answer to this I would say that this same test was given at the beginning of the school term to 270 pupils at West High School, Minneapolis. This school I believe to represent a normal situation. The pupils here had the same preliminary training as those at the University High School. The median score made by the West High School pupils on the test was 56. I do not know what procedure was followed in this school after the tests had been given, but undoubtedly a very similar improvement would have taken place if a procedure similar to the one used at University High School had been used.

2. Few school systems have this revised course of study which was spoken of at the beginning of this article.

In answer to this I would say that a close examination of recent Junior High School texts for seventh and eighth grades will reveal the fact that a large part of them is given over to simple drawings and constructions which bring into use the names of

geometric lines and figures, and acquaintanceship with geometric facts and terms as contained in the above test. In the ninth grade there appears to be developing a well organized course incorporating many more of the same type of facts through the formula and its applications, and through the notion of similarity and proportion. Unquestionably what is happening is that a consecutive and well organized course of general mathematics is being developed for these grades, of which the geometric phase is but one of the many features.

I believe, due to the progress that this revised course of study, outlined above, is making throughout the country, and due to the results of the test already referred to, that the following conclusions are warranted:

1. Pupils approach the study of plane geometry today with a large store of geometric information. Two or three weeks need no longer be spent in the teaching of names and terms. We may justly begin farther along in our work. Recent trends in reorganization would seem to indicate that this store of facts will be increased from year to year rather than decreased. Moreover, this information which the student possesses has been a gradual acquisition and is an integral part of his knowledge in a way that it never was when taught at the beginning of grade ten.

2. We should now be able to do considerably more work in the tenth year than was formerly possible. Our brighter pupils should very well be able to cover a course as outlined in the National Committee Report in less than one year. One of the following developments seems a logical consequence:

a. A combined course in plane and solid geometry for our brighter pupils in grade ten.

b. A course in plane geometry supplemented by considerable mensuration work in solid geometry.

c. A course in plane geometry supplemented by a considerable unit of trigonometry.

Some such revision will have to take place in grade ten or else I believe that we are not taking full advantage of the reorganization already effected and that is continually being effected in the grades immediately below the tenth.

MEETING THE ATTACKS ON ALGEBRA

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Spring Meeting, May 2, 1925 Association of Teachers of
Mathematics in New England

No subject in the high school curriculum, according to some educators, has yielded such unsatisfactory results as Algebra. It has been the subject of most severe criticisms on all sides, and has been held responsible for a great deal of freshman mortality. A leading educator of Massachusetts, in one of his public speeches, has expressed the desire that less time should be devoted to the study of algebra in the high school. An influential body of educators has even gone so far as to ask some of our Massachusetts colleges to reduce their entrance requirements in algebra. We not only find educators loudly declaiming algebra; but this spirit of criticism has carried still farther. We know that many school children hate algebra because their parents hated it, or because other pupils have told them how uninteresting and difficult it is. Parents dread the time when their children must study algebra. In a magazine article of recent date, a father was discussing the education of his daughter. In the course of the discussion he said that his daughter did not go to college because of her intense dislike of algebra.

What has been the result of all this? Algebra, except in the strictly college preparatory schools, has become more and more an optional subject. The majority of pupils now finish their high school course with no broader insight into mathematics than that obtained in their first eight years of school.

Perhaps some of these criticisms are justified, after all. We must admit that there has been general dissatisfaction with the results of teaching of algebra. It is claimed that algebra, as it has been taught, is too remote from life to interest the pupil and has outlived its usefulness as a subject of secondary school instruction. Modern psychologists claim that there is no such thing as mental discipline; therefore mathematics teaching as a whole has no value. I think that, although we are unwilling to do so, we must admit that traditional algebra has little educational value. If algebra consists chiefly in the acquisition of certain facts, in the manipulation of symbols, and in the solution

of a few book problems, it deserves to be the victim of propaganda. It must fight a losing fight and should be "pitched overboard," for there are no tangible results forthcoming from the algebra as taught in the old days.

It is our job as mathematics teachers to dispel this current idea that algebra is as useless as it is difficult. We must endeavor to define and to teach algebra so that it becomes interesting and educationally valuable to young people. We must present it in such a way that an average pupil can grasp it and reason it out as he goes along. Pupils must be taught to handle thoughts and not merely symbols of thought. We should forget the mechanical notions and see algebra through the eyes of the pupils, and relate the subject to them.

We can do a great deal toward dispelling the doubts as to the usefulness of algebra by the proper presentation of the subject matter. Organize what the pupil knows before introducing new facts. Each fact and process should be thoroughly mastered by the pupil before he is plunged into a new difficulty. Make Algebra real and interesting to the pupil right from the outset.

The real impetus to this movement of reform in algebra was given by the report of the National Committee on Mathematical Requirements in 1916. Since that time a wave of reform has been gradually gaining headway in mathematics, and the new algebra is steadily making progress in the high school curriculum. Many new textbooks have appeared recently, carrying out the recommendations laid down by this committee. These books reflect to a very high degree the present trend of thought. It seemed to me that it might prove interesting to make some comparisons between the presentation of topics in the traditional algebra and in the new algebra, and thus see what is being done to meet the attacks that have been constantly made on algebra.

The topics of Algebra as most of us were taught in high school were presented in some such order as the following: A short introduction explaining the use of symbols in algebra with a few equations to illustrate the use of symbols, explanation of the terms, factor, exponents, coefficients, parentheses, radical sign, etc.; a short exercise, as a rule, in evaluating algebraic expressions, in which the pupil was taught the correct order of fundamental arithmetical operations.

With this body of definitions and explanations as a background, the pupil was plunged immediately into the chapter called "positive and negative numbers." Then the mechanical rules for addition and subtraction of monomials were given with numerous examples. Following this the law of signs for multiplication and division was given. To be sure, some of these rules were explained, e.g.

$$-4 \text{ times } 3 \text{ means } (-4) + (-4) + (-4) = -12.$$

I wonder how many pupils really understand this explanation, whereas -4 times -3 means that -4 is to be subtracted 3 times, etc. Then addition of monomials and of polynomials was presented next. There followed a chapter on simple equations, which at first glance seemed to have no connection with the preceding material, although we later see that some of the equations involved the uniting of similar terms. Next came subtraction of polynomials, more equations, parentheses, multiplication of polynomials, more equations, but this time perhaps involving parentheses, division, more equations, etc. Each new topic was presented, but the need for it was not apparent. The pupil must learn a great many rules, then he is given problems and exercises in which to apply those rules. I do not mean to imply that the pupil should learn no rules in algebra; but the point we should make is that the pupil should first see the need for a process; this process is rationalized as far as is permissible in the reasoning power of the pupil. He understands the process and is ready for the rule. Certain processes gradually do become mechanical, but this mechanical handling of them is justified provided the pupil thoroughly understands what he is doing.

To go a little farther with the traditional algebra. In the first year work the pupil was expected to cover many cases of factoring, including the sum and the difference of cubes. The finding of the highest common factor by the Euclidean method was enough to discourage any pupil, especially when he could see no immediate use for it, for the fractions he dealt with should not be difficult enough to require this method of finding the highest common factor. I might continue thus and show how much was expected of the pupil, but I know this would only prove very tiresome.

As a contrast to this, let us examine part of a table of contents of one of the recent algebras.

I. Formulas—Equations.
II. Formulas of Mensuration—Approximate Computation.
III. Application of Formulas to Science and Industry.
IV. Linear Equations—Graphs.
V. Positive and Negative Numbers, etc.

Note the late introduction of positive and negative numbers. Let me quote just a little from the preface to indicate just what steps are taken to improve the teaching of algebra and to overcome so many of the objections to traditional algebra.

"The approach to algebraic symbolism and technique is through well-known formulas. Continued emphasis is placed upon rational methods of calculation and the determination of the reliability of numerical results when computed from data obtained by measurement. The simple and relatively late introduction of the negative number and the definite attempts to overcome the difficulty of handling of the signed number are some of the features of the books," etc.[1]

The "New Algebra" is built chiefly around the formula and the equation. The pupil is introduced to algebra through the formula, with which he is already somewhat familiar. The algebra is reorganized from the problem-solving point of view. The very first equations should be those that solve problems, for we certainly can not teach equations until we know how equations are obtained. Every new difficulty should be introduced by a problem, so that the pupil can discover for himself that he needs the new process to solve that particular problem.

Rather than go into detail concerning all the topics in the "New Algebra," I shall discuss the treatment of two which are presented in a manner very much different from that found in the traditional algebra. I refer to the solution of equations, and to the handling of problems.

The "Great Equation Law" as one author calls it is constantly emphasized throughout: "What ever we do to one side of an equation, we must also do to the other." The various types of equations are presented in the order of their difficulty; *e.g.*

I. $4x = 12$.
II. $3x - x = 16$.

[1] The above is taken from the Revised Edition of Junior High School Math. Third Course—by Vosburgh-Gentleman-Hassler.

Then $3x + 5 = 17$ in which we should like to find $3x$ first. But we have 5 more than $5x$. Therefore we subtract 5 from both sides of the equation, or $3x = 12$.

Perhaps some may ask how we can handle $2y - 25 = 43$ when we have not taken up negative numbers, or addition or subtraction. We should like to find $2y$. But we are short 25 of $2y$, i.e., we have a shortage of 25. To make up this shortage we add 25 to both sides of the equation, $2y = 68$. The minus sign indicates subtraction, or a shortage and that is all there is to it. The word transpose is never used. We teach the pupils a common-sense method of operation upon equations and not some mechanical process. It is important to note that the pupil will never have the occasion to divide by a negative number in solving these simple equations, because he makes up shortages if there are any. The question may come up: how shall we handle this: $2x = -6$. If $2x$ is a shortage of 6, then $1x$ or x is one-half of that shortage or -3. In other words, we treat quantities with minus signs before them as shortages. Thus we carry on our development of the equation dealing with those involving parentheses, and those involving fractions which have monomial denominators. We always introduce each new type by means of a problem and then give a common-sense way of arriving at an answer. No work should be accepted that is not carefully checked. To be sure, the old algebra encouraged checking. We do more than that: we insist on checking and accept no work as complete until it has been carefully checked. There are two more points I wish to indicate in connection with equations: A series of steps have been built up to aid in the solution of equations. The form is due to Mr. Evans of Charlestown.

In my own work with ninth grade classes, I have found these exceedingly helpful. Most of you are, no doubt, familiar with these:

(1) Perform any indicated multiplications.

(2) Unite similar terms in each member.

(3) If there are any terms with minus signs before them, add enough to make up these shortages.

(4) Subtract from each member the smaller unknown term.

(5) Subtract from each member any known term that stands beside an unknown term.

(6) Divide both members of the equation by the coefficient of the unknown.

The second point is the use of a convenient symbolism to enable the pupil to describe what he has done. This also, I believe, is due to Mr. Evans, and is an outstanding feature of the new Algebra. The pupil decides what he must do before he actually does it.

① $\frac{2x}{3} - \frac{2+x\ 1}{7} = x - 2$

②	$14x - 3\ (2x + 1) = 21x - 42$	① × 21
③	$14x - 6x - 3 = 21x - 42$	② same values
④	$8x - 3 = 21x - 42$	③ Same values.
⑤	$8x + 39 = 21x$	④ + 42
⑥	$39 = 13\ x$	⑤ — 8x
⑦	$3 = \quad x$	⑥ ÷ 13

Check

$$\frac{2x}{3} - \frac{2x + 1}{7} = x - 2$$

$$2 - 1 = 3 - 2.$$

Ans. $x = 3$.

The discussion of the equation is very sketchy, indeed; but I hope I have succeeded in showing how different its development is from that in the traditional algebra.

The study of mathematics should result in the development of power, rather than in the acquisition of facts and ability to handle difficult and intricate manipulations. There is no better way of developing that power than by the solution of problems. One of the common remarks we often hear from pupils is: "I like Algebra, but hate problems," as if problems were separate and distinct from algebra. We find model examples in most of the algebras solving problems, but the pupil is not told how to go about it in such a manner that he will be able to handle any problem that may come up in the course of his study. In "Everyday Algebra," a book to be published shortly by Houghton Mifflin Co., the author has used problems as the vehicle of introducing each new algebraic fact. He has set up certain steps in problem solving which are well worth considering at this point, for they again show what a decided step in advance the new algebra has taken to overcome the great dislike of problems.

I. The first step is algebraic analysis, i. e. decide what quantities are to be listed.

II. The second step is to express related quantities algebraically. Consider (1) which quantity will be represented by a single letter (2) how the other quantities are related to the quantity you have selected and (3) the total, which can be represented both algebraically and numerically.

III. The third step is to form an equation.

IV. Solve the equation.

V. Test the answer to see if it meets the condition of the problem.

He points out that there are three possibilities in setting up the equation:

(1) We have 2 abbreviations for the same quantity.

(2) We use some fact stated or implied in the problem, but not used in the list.

(3) We substitute in a formula.

There are various other topics in the new algebra, that are well worth considering, such as the late introduction of negative numbers and of algebraic addition and subtraction, approximate computation, extensive use of various kinds of graphs, etc., all of which are treated in a manner that indicates very clearly that mathematicians have taken up the cudgels in defense of algebra.

Some teachers may say: "Well those ideas are all very fine; but can we follow them out with pupils who are preparing for college? Have we the time to do it?" Our answer is, "Why not?" Shouldn't the pupils who are going to college also be trained to use their common sense and understanding in algebra? The College Board has indicated that it realizes and appreciates the present tendency toward improvements in the teaching of algebra by its willingness to reduce in its requirements a great deal of algebra dealing with algebraic technique and to omit certain topics previously required, such as the square root of a polynomial, the finding of the highest common factor as a separate topic, etc.

We, teachers of mathematics, are fortunate to be a part of the mathematical world at a time when such a tremendous reorganization is taking place. We have seen Algebra attacked on all sides with fatal results to its position in the high school curriculum. We now see the advent of the new algebra which bids fair to meet all these attacks and to restore algebra to the high position it deserves in education.

THE SOLUTION OF QUADRATIC AND CUBIC EQUATIONS ON THE SLIDE RULE

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Any quadratic equation may be expressed in the form

$$x^2 + bx + c = 0. \quad \ldots\ldots\ldots\ldots (1).$$

If the roots are real it is only necessary to find two real numbers which product is c and whose algebraic sum is $-b$. Hence to solve a quadratic equation of this type on the slide rule set the cursor at c on the A scale and move the slide until the sum of the numbers under the cross hair on the B scale and over the index on the A scale is equal to $-b$ these two numbers are the roots.

Example: (1) $x^2 + 5x + 6 = 0. \quad \ldots\ldots\ldots\ldots (2).$

Set the cursor at 6 on the A scale. Move slide to right until 3 on the B scale is under the cross hair: then the index is under 2 on the A scale. Therefore 3 and 2 are roots.

Example: (2) $x^2 - 3x - 10 = 0.$

Set the cursor at 3 on the A scale, move slide until 5 on the B scale is under the cross hair and the index is below 2 on the A scale. Then $+5$ and -2 are the roots.

Even if the roots are imaginary, they may still be found on the slide rule by means of the following lemma. The condition for imaginary roots in (1) is that $\frac{b^2}{4} - c < 0$. If now c' be made equal to $\left(\frac{b^2}{2} - c\right)$ then $\frac{b^2}{4} - c' = \frac{b^2}{4} - \left(\frac{b^2}{2} - c\right) =$

$$c - \frac{b^2}{4} > 0.$$

Therefore to transform a quadratic equation with imaginary roots into the corresponding one with real roots, multiply b by $\frac{b}{2}$ and subtract c. Having found the real value subtract $\frac{b}{2}$ and multiply by c. This will give the imaginary part of the complex root, the real part being $\frac{b}{2}$.

Example: Solve $x^2 - 6x + 15 = 0$. ....(3)

$$b \times \frac{b}{2} = \frac{36}{2} = 18, \quad 18 - 15 = 3.$$

Therefore the corresponding equation with real roots is

$$x^2 - 6x + 3 = 0. \quad \text{.................(4)}$$

The solution of (4) on the slide rule is 5.45 and .55. Subtraction of $\frac{b}{2} = 3$ gives 2.45.

Therefore the roots of equation (3) are

$$3 \pm 2.45\, i \quad \text{................(5).}$$

Runge[1] has shown how to use the slide rule with inverted slide to solve cubic equations of the form

$$U^3 + 5U = 3. \quad \text{................(5).}$$

But this is also solvable as follows without inversion of the slide. If we read u upon the C scale then u^2 is just above it on the B scale. Then the product $u(u^2 - 5) = 3$. Hence set the cross hair of the cursor at 3 on the D scale, move the slide to the right until the number u^2 on the B scale minus 5 is equal to the number on the D scale below the index of the slide. (Fig. 1.)

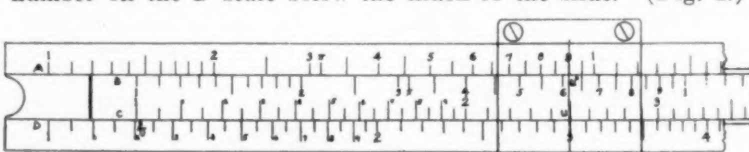

FIG. 1.

Again if u is negative

$$u^2 + \frac{3}{u} = 5$$

Hence move the slide to the right until the number under the cross hair on the B scale plus the number below the index of the slide on the D scale equals 5. Read $u = 1.84$. (Fig. 2.)

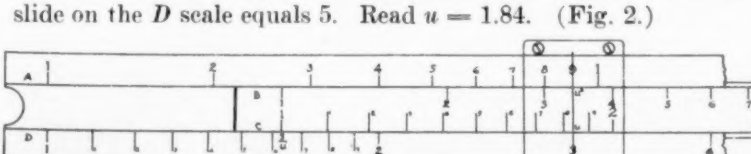

FIG. 2.

[1] Runge: Graphical Methods; p. 46.

There must also be another negative root. To find this move the slide to the left until as before $u^2 + \frac{3}{u} = 5$. Then $u = -.66$.

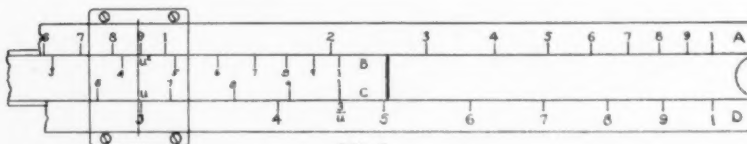

FIG. 3.

If the cubic is in the form $u^3 + au^2 = b$, it may be readily solved for one real value. Let us solve $x^3 + 3x^2 = 20$ or $x^2(x + 3) = 20$. Set the cursor to 20 on the A scale. Move the slide to the right until the number on the B scale under the cross hair minus 3 is equal to the number on the D scale below the index of the slide is the solution 2. (Fig. 4.)

FIG. 4

Another interesting solution of such an equation

$$x^3 + 3x^2 = 54 \text{ or } x + 3 = \frac{54}{x^2}$$

is: set the cursor at 54 on the A scale, then read x^2 on the B scale and x just below it on the C scale. The index of the slide will always point to $\frac{54}{x^2}$ on the A scale. Hence move the slide to the right until the number under the cross hair on the D scale plus the three of $(x + 3)$ is equal to the number $\frac{54}{x^2}$ on the A scale.

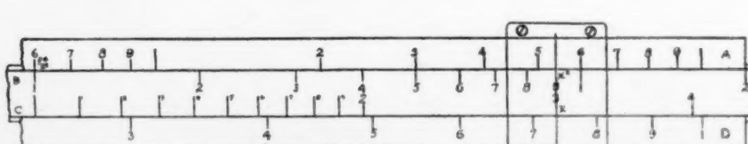

FIG. 5.

The following example shows a general method of solution in which all three roots of the cubic are found. If the cubic is in the form $x^3 - 5x - 3 = 0$ and if a, b, and c are the roots, $a + b + c$ must equal zero, since x^2 is missing: also $abc = 3$. Therefore, if the first root is found as in (Fig. 6)

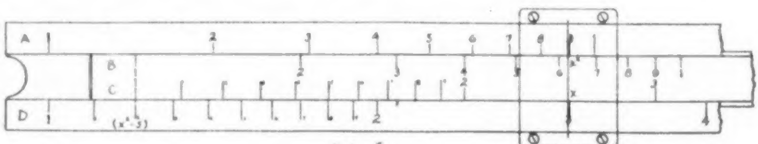

FIG 6

$$x = 2.5 = a. \quad \text{Then } b + c = -a = -2.5$$

$$\text{and } bc = \frac{3}{a} = \frac{3}{2.5} = 1.2$$

Accordingly having found the first root, move the cursor over to 1.2. Then move the slide to the left until the number under the cursor on the C scale plus the number below the index on the D scale will together equal -2.5. (Fig. 7.)

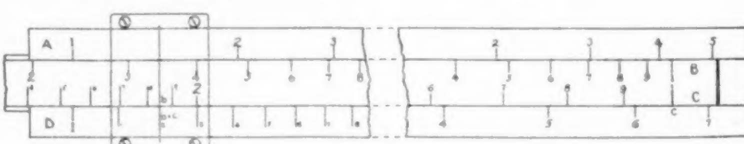

FIG. 7.

Complete solution $a = 2.5$, $b = -1.84$, $c = -.65$.

SOME INTERESTING SIDE LIGHTS UPON ELEMENTARY MATHEMATICS, INTRODUCING A NEW EXPANSION OF BINOMIAL THEOREM

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In this article I hope to call attention to certain diversions in the study of elementary mathematics as pursued in high schools today.

The value of any diversion in the study of a given subject is too well known to merit any particular comment here. For instance the craze in cross-word puzzles, which has recently swept over the country, has exerted a noticeable influence in word study, and although this stimulant may be short lived, its influence will undoubtedly be felt for some time in the increased use of the dictionary. While cross-word puzzles may have little or no bearing upon mathematics it serves, however, to illustrate the power of such diversion, in creating or reviving an interest in any particular thing.

We have a great many interesting and fascinating combinations in the study of mathematics and if such things are pointed out to the student at the proper time it may relieve a great deal of the humdrum during the study of this subject. It has been in following some of these combinations that I have stumbled upon some interesting facts.

In this article I wish to show some relations that exist between powers of numbers. Let us first look at the relations that exist between the squares of two numbers. I early learned that the difference between the square of two numbers is equal to the product of their sum by their difference. Let us take two numbers—say 5 and 8 for example.

Sum of the two numbers	equals 13
Difference between the two numbers	equals 3
Product of the sum and difference	equals 39
Now the square of 8	equals 64
Now the square of 5	equals 25
Difference between the squares	equals 39

It should prove refreshing to the student to try out many other pairs of numbers.

Now let us consider the cubes of the numbers. I have discovered that the difference between the cubes of two numbers is the product of the difference between the two numbers and the remainder obtained from deducting the product of the two numbers from the square of their sums. An illustration may make this clearer. Taking for example the same two numbers 5 and 8—

Sum of the two numbers equals 13
Square of the sum of the two numbers euals $13 \times 13 = 169$
Difference between the two numbers equals 3
Product of the two numbers $5 \times 8 = 40$.
Square of the sums minus the product $169 - 40 = 129$
$129 \times$ difference $= 129 \times 3 = 387$, the difference between the cubes of the two numbers.

$$8^3 = 512$$
$$5^3 = 125$$
$$\text{Dif.} = 387$$

This should also be tried out on several other combinations.

Now that these experiments have been made with the ratios that exist between the squares and cubes of numbers of one term let us try out the binomial.

The process used in elementary algebra is rather tedious and the well-known formula for the expansion of a binomial is also tedious, so I have used another system of expansion which can be made up in tabular form and may prove to be refreshing to the student if introduced at the proper time. The table which follows and is labeled Fig. 1 is made out for a binomial expanded to the 10th power. The student should continue this still further. In the table the numbers indicate the coefficient and it is understood that the exponent is one.

The method of constructing the table is quite simple, beginning at the top we start with a simple integer, on the line below we add one integer making the term a binomial of the first power—the next line or second power is obtained by starting at the left with one, the next term is obtained by adding together the two terms immediately above, etc., always adding one at the extreme right, each succeeding line being obtained in a like manner.

	1
1	1 + 1
2	1 + 2 + 1
3	1 + 3 + 3 + 1
4	1 + 4 + 6 + 4 + 1
5	1 + 5 + 10 + 10 + 5 + 1
6	1 + 6 + 15 + 20 + 15 + 6 + 1
7	1 + 7 + 21 + 35 + 35 + 21 + 7 + 1
8	1 + 8 + 28 + 56 + 70 + 56 + 28 + 8 + 1
9	1 + 9 + 36 + 84 + 126 + 126 + 84 + 36 + 9 + 1
10	1 + 10 + 45 + 120 + 210 + 252 + 210 + 120 + 45 + 10 + 1

Fig. 1.

Formula. $(a + b)^n$

$$a^n + na^{n-1}.b + \frac{n(n-1)}{1.2} a^{n-2}.b^2 + \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} b^3 + \ . \ . \ . + b^n$$

The use of the table is quite simple also after it has once been constructed, but it must be borne in mind when writing a power, that the first term must be arranged with the exponents in the descending powers and the second term with the exponents in the ascending powers. As an illustration let us find the sixth power of $(a + b)$ looking at the table and finding the form for the sixth power we proceed to write

$$a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6.$$

Now is that not easier than applying the formula or multiplying it out in five tedious operations?

In looking at the table one may find a great many interesting combinations, for instance it is noted that the figure takes the form of a triangle, a very stable form, that the two outside lines are ones, each of the second lines advance in arithmetical progression, i.e., they are consecutive numbers, that all the lines are advanced by adding to the last number on a given line the number above and ahead in the former line. In looking at Fig. 1 we see the fine dotted lines connecting numbers together, for instance in the second row on the left side we see $1 + 1 = 2$, in the third row $3 + 3 = 6$, and so on for each line throughout. A little observation will reveal a great many of these coincidents.

WHY IS IT?

P. STROUP
West High School, Cleveland, Ohio

I have just looked through copies of 10 current high school geometries and in not one of them can I find a direct and explicit proof of the fact that if the square on one side of a triangle equals the sum of the squares on the other 2 sides, it is a right triangle. In some books it is proved indirectly by proving its converse and its opposite, but no attention is called to the fact. Some authors use the fact in problems without having proved it or mentioned it and these same authors seem worried in their opening chapters if the converse of any little proposition is assumed to be true without giving the proof of it.

I suppose there are as many proofs of this fact as there are of its famous converse, but they turn out in practice to be somewhat trickier than their relatives. The easiest and most satisfactory one that I found is the converse of the first proof usually given. Let h^2 equal a^2 and b^2. Take a length c on h adjacent to a so that ch equals a . Call the rest of hd and then dh equals b^2. Hence c/a as a/h which proves two triangles similar as they have the included angle in common. Same with dh equal b^2. By the equal angles in the similar triangles one angle of the triangle can be proved to equal the other two.

9B ALGEBRA

The teaching of beginning algebra can be a real mental stimulus if the student is led into it through the door of experiment and discovery rather than driven in in close formation with the 'goose step.' Too many rules and too many exercises of problems of exactly one kind enables and encourages students to get by with a minimum of analytic and creative thinking, largely by mere remembering. Just another set of facts that are so because it says so in the book and the teacher says so. Students that are compelled to get algebra this way or not at all should not be taking it. They will not be able to use it or go on with its study. It is the presence of so many of these students that makes so many teachers feel compelled to drill it into them. It would be better for them if a real effort were made to get them to think.

As it is they are doped with rules and passed on to geometry or passed out of school. As for those students of sufficient mental caliber to appreciate for themselves the why and wherefore of algebra, it is not a square deal to them to be deprived of the opportunity to do so.

After years of trying to get algebra across in the order and manner presented in the algebras supplied, this suggestion gradually evolved. Present to the students only word problems and only take up such technical algebra as is necessary in the solution of those problems. Thus the immediate problem would supply the incentive for the study of the necessary algebra and give satisfaction in its use and the immediate relation between x and the value in the problem that it represented furnishes a less slippery handle to get hold of the difficulties of using letters to represent numbers. The student will see just where the difficulties arise that are not present in arithmetic. The necessity of quantities less than 0 is met right where it lives and can be studied in its natural habitat. The parenthesis comes in as a needed form of expression and not as another aggravation.

Back of this is the assumption that even in the algebra class the future citizen is the thing and not the little bit of algebra. How small this bit is, is appreciated by those who ask their students to use some of their year of algebra in the geometry or by the teacher of advanced algebra. Algebra presented in the way indicated would give a laboratory subject the equal of physics or chemistry in opportunity to experiment and draw and generalize conclusions. The rules of algebra such as 'like signs give plus' will have to be introduced as their use is a mental economy that cannot be ignored, but they should not be introduced until there is a background of personal experience which makes the rule more than mere words.

This experiment could not be tried out in its pure form in a regular school, but I have tried a modification of it. For over two months, using a current algebra, we ignored everything in the book except the word problems. The students worked at their seats in class and asked help by raising their hands. Anything of general interest was talked over at the board. At first the formation of the vocabulary and the translation into an algebraic equation got all the attention, but soon the problems of technical algebra appeared. The difficulties of the translation

were of course never completely conquered, but I see no reason for stepping around them on that account. Algebra will be useless to one who has not sufficient mental power to form the equation from the problem and the battle might as well be fought here as anywhere. There is strength to be gained in fighting and character to be gained in facing the issue.

The temptation to use two unknowns which suggested itself to the students quite naturally was resisted for a while, but in time we took up simultaneous equations and though I only showed them elimination of one letter by addition or subtraction, one boy discovered elimination by substitution and a boy and a girl eliminated two of three letters without any help. We then graphed equations for two weeks. From this we went to multiplication, oral and then long, then direct to long division, and if properly treated this furnishes one of the best exercises in algebra. The division is put in the same form as the multiplication, the divisor at the top and the dividend in place of the product and the suggestion is given that we are to find out what the divisor was multiplied by to give that product. With no other help some of the students fill in all the terms for the hardest problems in the exercise. Others must be reminded how columns cancelled out in multiplication and some columns had never more than one, two, or three terms. The only objection to this method is the guessing how many lines it is going to occupy but this is easily gotten around by leaving the product in the book or carrying it on the edge of another piece of paper until the place is reached for it. Remainders come in quite naturally. You will be surprised how the students take to this method and what a fine mental exercise it is. The mystery is all removed and they do with understanding what is a mere rule by the other method.

This method also has the advantage of leading directly into factoring and furnishes a general method for factoring from which the special methods can be gradually worked out. The students are simply turned loose on a fairly hard exercise in factoring with the suggestion that it is like division, but both of the expressions that were multiplied together are to be found. This furnishes a fine exercise in experiment and observation and the special methods will gradually be evolved. The special rules of multiplication are also met in this way and the connec-

tion between them and the factoring is more direct than by the other methods.

It would not be surprising if the students under such a system did not show the rapidity of advance in technical algebra that they would under the drill system. But I cannot but believe that they are better citizens and in the long run will make better algebricians. In encouraging a person to think it is desirable that there be presented some logical matter to think about in which there is no dispute about the conclusions. This is the value of algebra and geometry in the high school curriculum. Then this matter should be presented in a manner that permits the person not so much to see the truth as to draw conclusions for himself and discover the truth. This takes time and cannot be accomplished when the student is harried by constant drill. What he needs is personal attention to show him just how far his thinking is correct and just where it got off the track and is inconsistent with his other ideas.

AN APPROACH FOR 10B GEOMETRY

Although not expecting to find a royal road to geometry, the teacher of the beginning students is justified in seeking the most desirable path to the 'hinterland,' the path that has the most uniformly gradual assent to the heights. The path here outlined starts near the usual place of vertical angles, but follows angles through multisided polygons and delays the meeting with congruent triangles for nearly a month, in the meantime surmounting such obstacles as the common axioms and many shorter arguments.

The beginning student is assumed to know what he means by a straight line and the opening statement is that if two lines cross, they point in different directions and the difference in direction is an angle. Parallel lines are then defined as lines that point in the same direction. This leads to two conclusions, that parallel lines cannot meet and that if they are crossed by a third line the difference in direction or the angle must be the same at both crossings. Vertical angles are then proved equal, which makes available all the propositions about parallel lines cut by a transversal. The converses are left until the three angles of a triangle are proved equal to 180° and are then made to depend on that fact. Attention is called to the turning of a vehicle

by backing and going forward which is a practical proof that the three angles of a triangle equal 180. A vehicle going around a circle demonstrates the fact that the sum of the exterior angles must be 360 for any number of sides.

These facts permit the use of a long series of problems that demand reasoning with the simple axioms without introducing the longer proofs using congruent triangles.

After congruent triangles have been introduced and the three main cases proved, construction problems are more effective for their exercise than propositions. That is, taking the problem of bisecting an angle, the pupil is more impressed with what are called the given facts if they are true because made true by some other pupil or the teacher. There is more of reality to the problem if one student is asked to bisect an angle that the teacher has drawn and he or another student is asked to prove that what he has done will make the two parts necessarily equal. Other standard constructions can be used in this way and many propositions can be brought in. The various ways of constructing a square is a good exercise for the purpose. It cultivates observavation to be ruled out for saying something was done that was not done. The parallelogram can be introduced in this way. The order of the remaining propositions is not important.

This order of introduction of the beginning propositions is preferable to any that I have found. The 'pons asinorum' in the modern teaching of geometry is the argument of the equality of corresponding parts of congruent triangles. This is met ordinarily when all the newness of geometry is fresh upon the student and it is an advantage to put it off for a month. In this month, the teacher, the student and the argumentative method get better acquainted and congruent triangles can be assailed in better formation. The facts about the angles of polygons are a more impressive set of facts to the student than those ordinarily taken up first which are usually no news at all. Also, proving the proposition about the angles of a triangle so early relieves the awkwardness of the proof of several propositions that is so evident in some geometries.

It is inconvenient to try this without a book embodying the idea, but an experienced teacher who is not afraid of getting off the beaten path will find relish in trying any modification of it that may seem best.

A SUPPLEMENTARY PROJECT IN FUNCTIONAL GRAPHS

By J. S. GEORGES
University High School, Chicago

A supplementary project in Mathematics may be used effectively as a means of inspiring those students who display mathematical ability and show special interest in the subject, and of interesting those who take Mathematics as a required subject, but have not developed any particular interest in it. It offers to the latter a glimpse into the more pleasant aspects of the subject not ordinarily included in a text book, and to the former it opens new channels of mathematical interest, creates a desire for the acquisition of further and more extensive mathematical knowledge. The student who voluntarily, or at the suggestion of his instructor, undertakes to do a supplementary project in connection with his classroom work, is offered a splendid opportunity for the formation of the habit of independent thinking, and for acquiring the ability of reading comprehensively mathematical literature. Under proper guidance he lays the foundation for future research work.

Unfortunately the subject of supplementary project is not treated adequately in mathematical literature, in fact, it is hardly mentioned, and often the teacher who feels the need of supplementing his class-room work, whatever be his motive, is handicapped because of this lack of useful material. There is no organized material available. The few text books on general mathematics could hardly be said to furnish easy reading matter to the high school pupil who is just entering the threshold of Algebra or Geometry. And the help he may obtain from mathematical journals along this line is very limited. He is thus thrown upon his own resources. Nevertheless, if he has an adequate training in his subject, enthusiasm in his work, and willingness to promote the welfare of his class in Mathematics, he will find the field of Mathematics rich and varied in useful material for supplementary project work. Many classical problems of the elementary type, original and specially devised problems, or even problems of more advanced types, whose solution is not too difficult by means of the already accumulated information, may be used to advantage.

The subject of graphs, that is, the graphical representation of the functional concept, furnishes a rich field of investigation of the elementary type, which is both interesting and instructive to the student desiring to do supplementary work. The graphical representation of numerical facts and relationships by means of the geometric line segments and curves is important in Mathematics and should be emphasized, and at the same time is also very interesting to the child. The notion is not altogether new to him, having become familiar with it from newspapers and magazines, and the principles underlying this representation are readily understood and are tacitly assumed without much difficulty. It may be used by the beginner who is familiar with the bar graph or the continuous line graph to represent unrelated or related facts. Later on after the study of the equation and the formula, the graph is still of vital importance in the interpretation of the linear, quadratic, or cubic equation. Drawing of pictures is especially interesting to the child, and this geometric picturing of the algebraic notions not only keeps up his interest in the subject, but also impresses upon his mind the true union of these two apparently separate branches of Mathematics. Historically this fusion of Algebra and Geometry is of special interest to the mathematician and the student should not wait until he is in college or doing graduate work to appreciate the full significance of it.

The following project worked out by a junior high school pupil should suggest the type of supplementary work possible in this field. The project was assigned to six students at the end of the unit on Functions, which takes up direct and inverse variation, the functional representation, and evaluation of functions both by substitution and by the synthetic division, and the functional graph. The text book used was Breslich's "Junior Mathematics," Book III. All of the pupils were very enthusiastic in their work and seemed to derive a great deal of pleasure in their ability to trace new and advanced curves. The one reproduced here is selected for its brevity and it is typical of the kind of work handed in.

Subject: The Graphs of Cubic Functions.
Name: Robert E. Ascher, Junior Mathematics, Course III.
Instructor: Mr. Georges.

The graph of the linear function, such as $f(x) = x + 5$, is always a straight line, while that of a quadratic function, such as $f(x) = 2x^2 \times 3$, is a parabola. The graph of the cubic function is generally a curve which crosses the x-axis in three places. In this paper some unusual cubic curves are produced which differ in shape from the general cubic.

In graphing a cubic function usually nine points are needed. First make a table with arbitrary values for x and the corresponding values for $f(x)$. The values of $f(x)$ may be found either by substituting the value of x in the expression, or by using the method of synthetic division. Next select a set of axes and choose a convenient unit to plot the pairs of values. Finally join the points together with a "smooth" curve.

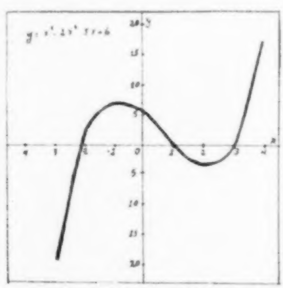

Fig. I

Figure I is the graph of the cubic function $y = x^3 - 2x^2 - 5x + 6$. It intersects the x-axis in three points, at $x = 1$, $x = 3$, and $x = 2$, which are the solutions of the equation $x^3 - 2x^2 - 5x + 6 = 0$.

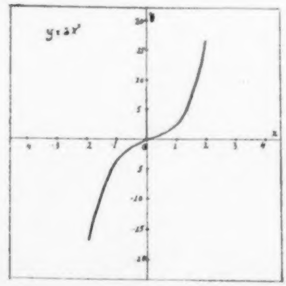

Fig. II

Figure II is the graph of $y = 2x^3$. The curve crosses the x-axis only once, at $x = 0$. The three points are brought together. This curve is called "cubical parabola."

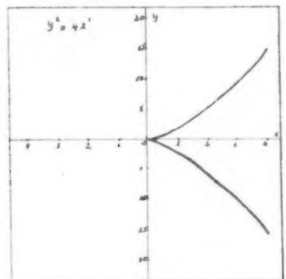

Fig. III

Figure III is the graph of $y^2 = 4x^3$. The graph of this cubic function is of an entirely different shape. All of the curve is to the right of the y-axis. It has a sharp point at the origin. It is called "cuspidal cubic." We can't use negative values for x for we can't extract the square root of negative numbers.

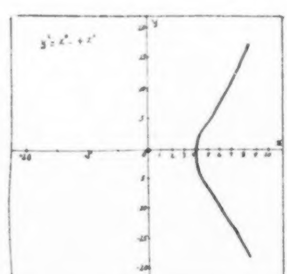

Fig. IV

Figure IV is the graph of $y^2 = x^3 - 4x$. This is still another type of cubic function and there is one place on the graph (at origin) where it does not connect with the rest of the graph at all. The point at the origin is called a "conjugate" point. It intersects the x-axis at $x = 4$.

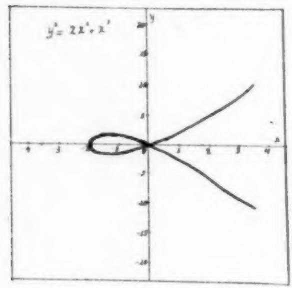

Fig. V

Figure V is the graph of $y^2 = 2x^2 + x^3$. This curve makes a loop with the knot at the origin. It intersects the x-axis at the origin and at $x = -2$. This curve is called "nodal cubic."

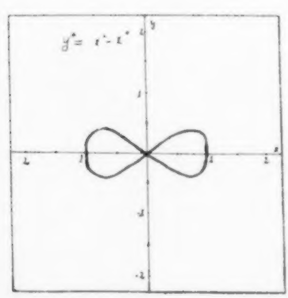

Fig. VI

Figure VI is very interesting. It is the graph of a quartic function $y^2 = x^2 - x^4$. The curve is like the figure eight. It crosses the x-axis at $x = \pm 1$ and the origin.

THE OLDEST ALGORISM IN THE FRENCH LANGUAGE

ANNA ELIZABETH HOUGHTALING AND
FRANCES MARGUERITE CLARKE

Introductory Note

The oldest French Algorism[1] known to be extant is in the Bibliothèque Sainte Geneviève, Paris. There is in the Bibliothèque Nationale another manuscript of a very similar nature but of a later date. Both are of unknown authorship and both are of the thirteenth century.[2]

The former work consists of folios 150, r-151, r;[3] the latter, of folios 154, r- 155, r. Since the two are similar in content, familiarity with the latter facilitates the reading of the former, for the French of the manuscript in the Bibliothèque Nationale is more modern in style. In spite of the fact that the two versions are available, the translation of either is exceedingly difficult in parts, the author being only vaguely familiar with his subject and writing so confusedly that the maze seems in one instance well-nigh hopeless.

There are four divisions of the algorism; (1) a treatment of numbers, (2) a consideration of the six fundamental processes, (3) division by the galley method, and (4) extraction of cube roots. The last two are not very clear and are not wholly correct.

Since no version of the manuscript seems to exist in any modern language (the French being far from the present form), and since the work is historically so important in the history of education as well as the history of arithmetic, the following trans-

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A paper prepared in Professor David Eugene Smith's Practicum in The History of Mathematics, (1924-25).

[1] Mohammed ibn Musa al-Khowarizmi, mathematician of Bagdad, *c* 835 A. D., wrote a treatise on arithmetic. Robert of Chester or Adelard of Bath translated and gave it the title of "Algoritmi de Numero Indorum" (Algoritmi was the nearest Latin spelling of the name al-Khowarizmi, meaning the man from Khwarezm). The various forms of algorism, augorism, etc., derived from Algoritmi, were applied to any study of numbers. Smith, *History of Mathematics,* I, 170; II. 9.

[2] See the Boncompagni *Bulletino* XV, 53, *and the Bibliotheca Mathematica,* IX (3), 60. The date of a manuscript is determined by the handwriting. This is not difficult since the handwriting is distinctive for the period.

[3] When medieval manuscripts were bound, only the folios were numbered on what we would call the right-hand page. The right-hand page of folio 150, for example, is commonly referred to as "150 *r*" (recto), the reverse side of the leaf being referred to as "150 *v*" (verso).

lation has been made. It is as literal as circumstances seem to warrant, but, as already stated, the author himself was so confused in one or two places that no translation can serve to show what he could have meant. The notes are intended to clarify the reading as much as possible, but in some instances offer only a possible solution to the author's thought.

Translation

Algorism is any representation using such symbols as the figures 9, 8, 7, 6, 5, 4, 3, 2, 1. The first is 1, the second 2, the third 3; and so on for the others up to the last which is called the cipher 0.[1]

Any of the numbers written in the unit's place has its original value; in the ten's place, ten times its original value; and in each succeeding place, a value ten times greater than in the place preceding. The cipher has no intrinsic value, but attached to any number it indicates a multiple of that number.

There are three types of numbers,—the digit, the article, and the composite. Numbers from 1 to 9, inclusive, are called digits; 10 and multiples of 10 are called articles; and combinations of articles and digits, such as 11, 12, 13, etc., are termed composites. If a number is a digit, it is written alone; if an article, the cipher is written first[2], and then the article; if a composite, the digit first, and then the article.

There are six parts to algorism: addition, subtraction, duplation, mediation, multiplication, and division. In addition or subtraction, begin at the right; in duplation, multiplication, or division, begin at the left.

In addition, write the greatest number above and the lesser below so that the first figure of the smaller number is beneath the first figure of the greatest number; then write the other figures in similar order. If there are more figures following, add the first two. If the sum is a digit, write it in line with the figures above; if an article, write the zero separately, and then write the article with the figures above in order.

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[1] Writing numbers from right to left is due to the Arabic influence.

The name cipher is apparently from the Sanskrit *sunya* (void), through the Arabian *sifr*.

The zero (0) itself appeared in India at least as early as the 9th century and is probably of Hindu origin. Smith, *History of Mathematics*, II, 69, 71.

[2] Beginning at the right.

Division should not result in a quotient exceeding 9.[1] You get one number for each time you divide. Place it beneath the first figure of the number beneath your number; then rewrite the figures of the lower number until the upper is greater than the lower,[2] finally dividing the number above, as you did before. Write the number which indicates how many times the upper number is divided by the lower.[3]

When in division the number above is less than the number beneath,[4] place this number elsewhere.[5] Prove the work by multiplication. If the division is correct, multiplication of the divisor, and addition of the number (remainder) will give the first number.[6] This proves the accuracy of the work.

Any number multiplied by itself [cubically] is a cube. To find the cube [root] of any square[7] number, first write the number, then under the last figure of a thousand write a digit that multiplied by itself cubically, destroys (with respect to this number underneath) the number above as much as possible.[8] Cut it off,[9]

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[1] That is, no figure in the quotient can exceed nine.

[2] This is easily seen in the galley method of dividing, then in use.

[3] To divide 1,140 by 95 write 95 beneath the 1,140.

Ninety-five is contained in 114 one time. Therefore 1 is the first figure in the quotient. Then $1 \times 9 = 9$. This is subtracted from the 11, and the 2 (being the remainder) is placed above the 11. Then $1 \times 5 = 5$. This is subtracted from 24.

$$\begin{array}{r} 1140 \\ 95 \\ \hline \end{array}$$

The 19 is placed above the 24. Then the 95 is written as shown. Since 95 is contained in 190 twice, 2 is the second figure in the quotient.

$2 \times 9 = 18$ $\qquad$ $19 - 18 = 1$

$2 \times 5 = 10$ $\qquad$ $10 - 10 = 0$

$$\begin{array}{r} 1 \\ 29 \\ 1140 \\ 955 \\ \hline 9 \end{array}$$

[4] To divide 1,230 by 95, the work would appear as follows: The remainder is 90. (The number above is less than the number beneath.)

$$\begin{array}{r} 9 \\ 380 \\ 1230 \\ 955 \\ \hline 9 \end{array}$$

[5] That is, move it one place to the right.

[6] That is, quotient times the divisor, plus the remainder, equals the dividend.

[7] The writer meant "cubic."

[8] For example, to find the cube root of 2197: Write 2197; under 2, the figure at the left, write a number whose cube, when subtracted from 2, gives a number as nearly zero as possible. In this case subtract 1 (the number is really 10). Cubing the 1 and subtracting the result from 2 leaves 1197.

[9] Cut it off means that from $a^3 + 3a^2b + 3ab^2 + b^3$ we tage away a^3.

treble it,[1] and place it two points in advance.[2] Now place the sub-treble below the treble.[3] Then place a new digit in front of the treble[4] which with the sub-treble multiplies the treble; then multiply the sum resulting by itself, and subtract it from the treble.[5] Take the digits by themselves cubically and subtract from the digits.[6] Cut off the digit,[7] treble it and place it two points in advance. Finally take a new digit which with the sub-treble multiplies the treble, and then square this sum, and subtract it from the last. Treble the digit, cube it, and subtract it from the digit. Continue the process thus, to the end. The number so operated on is a cube. The root is the sub-treble and the last digit found; multiply this by itself cubically and you will have again the original number. If there is any remainder the proposed number is not a cube, but you have the greater beneath. Cubing the root and adding the remainder will give the original number. If in trebling the digit no articles are formed, follow the rules of duplation.[8] If work with the trebled number is impossible, place a cipher for the digit; then write it and work as before. If there is only a cipher place a 0 for the digit; subtract it and place a 0. Place a 0 under the digit, and a 0 in line with the sub-treble. If this number whose root is to be taken has but two or three figures, at the right under the first figure, place a digit that multiplied by itself cubically destroys the number above as much as possible. The digit is then the root of the number.

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[1] To treble is to multiply it by 3. The process is really to take $3a^2$.

[2] To place it two points in advance means that, since we really have 3×10^2, instead of 3×1, we have not 3, but 300, the 3 being two more places to the left than in the case of 3.

[3] To place the subtreble below the treble seems to mean that we are to place $3a$ after (below numerically) the $3a^2$.

[4] That is, by dividing find b and place this (the "new digit") before (to the right of, algebraically expressed) the $3a$, giving $3a^2 + 3ab$., The rest of the statement probably means that he multiplies $3a^2 + 3ab + b^2$ by b, as in algebraic cube root; but it is evident that he does not understand the process.

[5] This would seem to mean that $3ab + 3ab^2 + b^3$ is now subtracted from the cube.

[6] The cube of the first part is subtracted from the given cube.

[7] Apparently he now repeats the process in order to find the third figure.

[8] This seems to mean that duplation may be used in case of trouble in multiplying, but the expression as it stands is meaningless.

The author thus abruptly ends his treatise. Although it is evident that he was hopelessly confused in some parts of this manuscript, especially the latter, it is worthy of consideration, first because of the unique place it holds as the oldest French treatise on algorism; secondly because it offers an interesting example of the early use of one as a number and also the use of the French word "assembler" for our process of addition.

A MATHEMATICS TEACHER'S CREED

I believe in oral work, and in lots of it.

I believe in blackboard demonstrations, by the teacher and the class.

I believe in anticipating difficulties and in forestalling them.

I believe in the inductive-development method of teaching.

I believe in motivation.

I believe in making encouraging remarks very frequently.

I believe in preparing and using devices.

I believe in competitions, especially in competing against oneself.

I believe in giving marks rather than in taking marks from pupils.

I believe in real, concrete problems.

I believe in problems without numbers.

I believe in frequent drills in all grades in the fundamentals.

I believe in giving pupils helps to remember.

I believe in making reasonable statements in problems.

I believe in correcting every bit of written work done by the pupils.

I believe in teaching without a text-book whenever possible.

I believe in supervising the pupils' work as they do it.

I believe that the slow pupils need not work as many examples as the fast workers.

I believe in cultivating the habit of checking all solutions.

I believe in estimating all answers.

I believe in approximations.

I believe in writing the "data" and the "required" in difficult problems.

I believe in the simplest method of solution of all problems.

I believe in rapid work.

W. P. Percival,
Macdonald College, Canada.

PERIODICALS

"What New York City is Doing for Dull-Normal Pupils." Ralph E. Pickett, School of Education, New York University. The Chicago Schools Journal, January, 1926. (Read before the National Vocational Guidance Association at Cincinnati, February, 1925.)

The author refers to the work of Seward Park Junior High, Textile High School, and the Manhattan Trade School for Girls, where some definite progress seems to be made in fulfilling the obligation of the schools with respect to those who are usually crowded out through failure and who often become a menace to society.

The dull-normal pupil is roughly defined as one who "is too dull to be called of normal intelligence and yet too normal to be called a moron, an imbecile, or something worse,—one whose I. Q. is probably between 75 and 95." It is for this type of pupil that we have either sapped our course of study of its vitality, or we have assumed that we ought not try to educate him.

While the highest type of academic education is suited to the training of the stronger pupils, as a preparation for leadership in life, the author urges, "Let us disabuse our minds of the tradition that the only real education is academic education, and that only those who can profit by such have a right to be educated. Let us realize our duty to all the children and let us avoid in future the waste of a useless expenditure of millions of dollars and countless precious hours in an attempt to force a mass of children to master that which they cannot master and ought not to be expected to master."

A. D.

The History of Arithmetic by C. L. Karpinski. Rand, McNally & Company, 1925. Pp. 200 + xii. Price $2.00.

The purpose of this book is to provide the teacher of arithmetic with a short, readable discussion of the topics she teaches. The volume begins with a description of systems of numerals other than our own, and the form that arithmetic operations assumed in the case of certain of them. The history of Hindu Arabic

numerals is then introduced, and, after a digression into the subject of notable textbooks in arithmetic, the development of arithmetic operations with these numerals is treated. Subsequent chapters deal with fractions, business arithmetic, terminology, denominate numbers, and the teacher and the teaching of arithmetic.

The work is well illustrated and the author has been clever in calling the reader's attention to salient points in the cuts in the notes that accompany them.

The chapter on textbooks in arithmetic includes a biliography of works printed in America prior to 1800. This is especially useful in that the libraries where these volumes may be found are listed.

The book includes much information that should be of value to the teacher of arithmetic, but it is disappointing in certain respects. Professor Karpinski tends to make assertions without giving the material needed to make his conclusions of the greatest use to his readers. For example, under the caption "History reflected in arithmetic," the author states that the textbooks printed in the Revolutionary and Civil War periods are especially rich in showing the way contemporary conditions were utilized in problem material, but, instead of illustrating this by concrete instances, he passes to a discussion of the fact that the textbooks printed in Colonial America reflect the nationality of the various groups of settlers in the languages in which they appear. This last point is well taken and well substantiated, but the bibliography that follows will be useful to but few, whereas a fuller treatment of the earlier point would be useful to many.

Such a work is informative rather than controversial, yet when Professor Karpinski says that the Roman pound sometimes contained sixteen ounces, it would be interesting to know the source of his information. Where his spelling of proper names departs from that given in such standard works as Professor David Eugene Smith's two volume *History of Mathematics*, it would be desirable to know the reasons that prompted this usage.

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